27/03/23

Maths Syllabus

(i) Differential Calculus.
(ii) Advanced differential calculus
(iii) Infinite Series.
(iv) Solid Geometry [Coordinates (3d), shortest dist. ~~dist~~. D.Cs, D.Rs, Planes cone, Cylinder, Sphere]

• **Hyperbolic Function** :- sinh x
cos h x.

is ↓ defined as

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\text{cosech } x = \frac{2}{e^x - e^{-x}}$

$\text{sech } x = \frac{2}{e^x + e^{-x}}$

$$\tanh x = \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

$$\Rightarrow \boxed{\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}}$$

$$\Rightarrow \boxed{\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}}$$

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$\boxed{\cosh^2 x - \sinh^2 x = 1}$, to prove L.H.S. = R.H.S

Proof :- L.H.S.

$\because \cosh x = \frac{e^x + e^{-x}}{2}$

$\therefore \cosh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2$

$= \frac{e^{2x} + e^{-2x} + 2 \cdot e^x \cdot e^{-x}}{4}$ — (i)

& $\sinh x = \frac{e^x - e^{-x}}{2}$

$\therefore \sinh^2 x = \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + e^{-2x} - 2e^x \cdot e^{-x}}{4}$ — (ii)

Adding (i) & (ii)

$\therefore \cosh^2 x - \sinh^2 x = \frac{e^{2x} + e^{-2x} + 2}{4} - \left(\frac{e^{2x} + e^{-2x} - 2}{4}\right)$

$= \frac{e^{2x}}{4} + \frac{e^{-2x}}{4} + \frac{1}{2} - \frac{e^{2x}}{4} - \frac{e^{-2x}}{4} + \frac{1}{2}$

$= 1 =$ R.H.S. proved

Hence, L.H.S = R.H.S. proved

$\Rightarrow \boxed{\cosh^2 x = 1 + \sinh^2 x}$

$\boxed{\operatorname{sech}^2 x + \tanh^2 x = 1}$

$\Rightarrow \boxed{1 - \tanh^2 x = \operatorname{sech}^2 x}$

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Differentiation of Hyperbolic function

Success diff

(i) $\frac{d}{dx}(\sinh x) = \cosh x$.

(ii) $\frac{d}{dx}(\cosh x) = +\sinh x$ ⇒ proof: Varified

(iii) $\frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right) = \frac{1}{2}\left[e^x - e^{-x}\right) = \sinh$

(iv) $\frac{d}{dx}(\tanh x) = \text{tan}\ \text{sech}^2 x$.

(iv) $\frac{d}{dx}(\text{sech}x) = -\text{sech}x.\tanh x$.

(v) $\frac{d}{dx}(\text{cosech}x) = -\text{cosech}x.\coth x$.

(vi) $\frac{d}{dx}(\coth x) = -\text{cosech}x$

(vii) $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$

(viii) $\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}$

(ix) $\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$

(x) $\frac{d}{dx}(\coth^{-1}x) = \frac{1}{1+x^2}$

(xi) $\frac{d}{dx}(\text{sech}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$

(xii) $\frac{d}{dx}(\text{cosech}^{-1}x) = \frac{-1}{x\sqrt{1+x^2}}$

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(1) If $y = ax^2 + bx + c$, then find successive diff =?

⇒ Soln:-

$$\frac{dy}{dx} = y_1 = 2ax + b.$$

$$\frac{d^2y}{dx^2} = y_2 = 2a$$

$$\frac{d^3y}{dx^3} = y_3 = 0$$

$y = f(x)$

$y_1 = f'(x) \to y' \to \frac{dy}{dx}$

$y_2 = f''(x) \to y'' \to \frac{d^2y}{dx^2}$

$y_3 = f'''(x) \to y''' = \frac{d^3y}{dx^3}$

$y_n = f^n(x) \to y^n = \frac{d^ny}{dx^n}$

(2) Find y_3 if $y = \log x$.

⇒ Soln:- $y_1 = \frac{1}{x}$.

$y_2 = -\frac{1}{x^2}$.

$y_3 = (-1)\frac{(-2)}{x^3} = \frac{2}{x^3}$. Ans

(3) Find y_n if (i) $y = \frac{1}{(ax+b)^m} = (ax+b)^{-m}$.

(ii) $y = (ax+b)^m$

⇒ Soln:- (i) $\frac{dy}{dx} = y_1 = -m(ax+b)^{-m-1} a$

$y_2 = (-m)(-m-1)(ax+b)^{-m-2} a^2$.

$y_2 = (-1)^2 m(m+1)(ax+b)^{-(m+2)} a^2$.

$y_n = m$

$y_n = m(m+1)(ax+b)^{-(m+n)} \cdot a^n$.

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$$y_n = m(m+m-1)\,a^n (ax+b)^{-(m+n)}$$ Ans

(ii) $y = (ax+b)^m$.

Soln:- $y_1 = m(ax+b)^{m-1} a$.

$y_2 = m(m-1)(ax+b)^{m-2} a^2$.

$y_3 = m(m-1)(m-2)(ax+b)^{m-3} a^3$.

$$y_n = \{m(m-1)(m-2)\cdots(m-n+1)\}(ax+b)^{m-n} a^n$$

(i)

$y_3 = (-m)(-m-1)(-m-2)(ax+b)^{(-m-3)} a^3$.

$$y_n = \{(-m)(-m-1)(-m-2)\cdots(-m-n+1)\}\times (ax+b)^{(-m-n)} a^n.$$

$$\Rightarrow y_n = \{(-1)^n\{m(m+1)(m+2)\cdots(m+n-1)\}\frac{1}{(ax+b)^{m+n}}\times a^n$$ Ans

In que. (1),

put $x = 1$., we get

$$\Rightarrow \boxed{y_n = (-1)^n \{1(2)(3)\cdots n\} \frac{1 \times a^n}{(ax+b)^{n+1}}}$$

$$\Rightarrow \boxed{y_n = \frac{(-1)^n \lfloor n \; a^n}{(ax+b)^{n+1}}} \quad \text{— (B)}$$

Put $a = 1$ & $b = 0$ in (B).

$$\boxed{y_n = \frac{(-1)^n \lfloor n}{x^{n+1}}} \longrightarrow \text{(C)}$$

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For try (H.W.)

H.W.

(i) Find y_{n+1} when $y = \log x$

Put $a = 1$ in above eqn (B), then

$$\boxed{y_n = \frac{(-1)^n \lfloor n}{(x+b)^{n-1}}}$$

(ii) Find y_n if $y = \frac{1}{x^2+a^2} = (x^2+a^2)^{-1}$.

$\Rightarrow$ Soln:- $y_1 = (-1)(x^2+a^2)$

(iii) Find y_n if $y = \sin x$.

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H.W.

(ii) $y = \frac{1}{x^2+a^2} = (x^2+a^2)^{-1}$, $y_n = ?$

Soln:-

$\Rightarrow y_1 = (-1)(x^2+a^2)^{-2}(2x)$.

M-1

$\Rightarrow y_2 = (-1)(-2)(x^2+a^2)^{-3}(2x)(2x)$ ✗

$\Rightarrow y_3 = (-1)(-2)(-3)(x^2+a^2)^{-4}(2x)^3$

1 1 1 1 1

$$y_n = (-1)^n \{1.2.3\ldots n\} \{(x^2+a^2)^{-(n+1)}\}(2x)^n$$

Ans

H.W.

(iii) $y = \sin x$, $y_n = ?$

⇒ Soln:-

$y_1 = \cos x$.

$y_2 = -\sin x = \cos\left(\frac{\pi}{2}+x\right)$

$y_3 = -\cos x = -\sin\left(\frac{\pi}{2}+x\right) = \cos\left(x+\frac{2\pi}{2}\right)$

$y_4 = \sin x\left(x+\frac{2\pi}{2}\right) = \cos\left(x+\frac{3\pi}{2}\right)$

V.V.I

! ! !

$$y_n = \cos\left(x+\frac{(n-1)\pi}{2}\right)$$

Ans

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(i) $y_{n+1} = ?$, $y = \log x$.

⇒ Soln:- $y_1 = \frac{1}{x}$.

$y_2 = -\frac{1}{x^2}$

$y_3 = \frac{(-)1(-2)}{x^3}$

$y_n = \frac{(-1)^{n-1}\{1, 2, \ldots (n-1)\}}{x^n} = \frac{(-1)^{(n-1)}(n-1)}{x^n}$

$$\boxed{y_{n+1} = \frac{(-1)^n \lfloor n}{x^{n+1}}}$$ ✓

(VVI) (ii) Method 1 $y = \frac{1}{x^2+a^2}$, $y_n = ?$

⇒ Soln:- $\because$ $y = \frac{1}{x^2+a^2} = \frac{1}{x^2 - i^2a^2} = \frac{1}{(x+ai)(x-ai)}$

$\Rightarrow y = \frac{1}{2ai}\left[\frac{2ai}{(x+ai)(x-ai)}\right] = \frac{1}{2ai}\left[\frac{ai+ai}{(x+ai)(x-ai)}\right]$

$\because i^2 = -1$

$\Rightarrow y = \frac{1}{2ai}\left[\frac{ai(x+ai) - x+ai}{(x+ai)(x-ai)}\right]$ $\because (a^2-b^2) = (a+b)(a-b)$

$\Rightarrow y = \frac{1}{2ai}\left[\frac{(x+ai)}{(x+ai)(x-ai)} - \frac{(x-ai)}{(x+ai)(x-ai)}\right]$

$\Rightarrow y = \frac{1}{2ai}\left[\frac{1}{(x-ai)} - \frac{1}{(x+ai)}\right]$

∴ By using the formula $y = \frac{1}{x+b}$, then $y_n = \frac{(-1)^n \lfloor n}{(x+b)^n}$

we get,

$$y_n = \frac{i}{2ai}\left[\frac{(-1)^n \lfloor n}{(x-ai)^{n+1}} - \frac{(-1)^n \lfloor n}{(x+ai)^{n+1}}\right]$$

$$= \frac{(-1)^n \lfloor n}{2ai}\left[\frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}}\right] \quad \text{---(i)}$$

Now,

Put $x = r\cos\theta$ & $a = r\sin\theta$,

squaring & adding, we get

$$(x^2+y^2)^{\frac{1}{2}} = r^2 \cdot 1$$

$$(x^2+y^2)^{\frac{1}{2}} = r$$

Also, $\frac{a}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$.

$$\Rightarrow \theta = \tan^{-1}\left(\frac{a}{x}\right).$$

$$\Rightarrow y_n = \frac{(-1)^n \lfloor n}{2ai}\left[\frac{1}{(r\cos\theta - ir\sin\theta)^{n+1}} - \frac{1}{(r\cos\theta + ir\sin\theta)^{n+1}}\right]$$

$$\Rightarrow y_n = \frac{(-1)^n \lfloor n}{2ai}\left[\frac{1}{(x-ai)^{n+1}} - \frac{1}{}\right]$$

$$\Rightarrow y_n = \frac{(-1)^n \lfloor n}{2ai\,(r)^{n+1}}\left[\frac{1}{(\cos\theta - i\sin\theta)^{n+1}} - \frac{1}{(\cos\theta + i\sin\theta)^{n+1}}\right]$$

$$\Rightarrow y_n = \frac{(-1)^n \lfloor n}{2ai\,(r)^{n+1}}\left[(\cos\theta - i\sin\theta)^{-(n+1)} - (\cos\theta + i\sin\theta)^{-(n+1)}\right]$$

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By De Moivre Theorem.

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

$$\Rightarrow \gamma_n = \frac{(-1)^n \lfloor n}{2ai\,(\gamma)^{n+1}} \left[\left(\cos\{-(n+1)\}\theta - i\sin\{-(n+1)\}\theta \right) - \left(\cos\{-(n+1)\}\theta + i\sin\{-(n+1)\}\theta \right) \right]$$

$\because \left[\cos(-\theta) = \cos\theta \right.$
$\left. \sin(-\theta) = -\sin\theta \right]$

$$\Rightarrow \gamma_n = \frac{(-1)^n \lfloor n}{2ai\,(\gamma)^{n+1}} \left[\{\cos(n+1)\theta + i\sin(n+1)\theta\} - \{\cos(n+1)\theta - i\sin(n+1)\theta\} \right]$$

$$\Rightarrow \gamma_n = \frac{(-1)^n \lfloor n}{2ai\,(\gamma)^{n+1}} \left[\cos(n+1)\theta + i\sin(n+1)\theta - \cos(n+1)\theta + i\sin(n+1)\theta \right]$$

$$\Rightarrow \gamma_n = \frac{(-1)^n \lfloor n}{2ai\,(\gamma)^{n+1}} \left[2i\sin(n+1)\theta \right]$$

$$\Rightarrow \boxed{\gamma_n = \frac{(-)^n \lfloor n}{a(\gamma)^{n+1}} \left[\sin(n+1)\theta \right]} \quad \text{(ii)} \qquad \left[\because a = \gamma\sin\theta \Rightarrow \frac{a}{\gamma} = \sin\theta \Rightarrow \frac{a}{\sin\theta} = \gamma \right]$$

$$\therefore \gamma_n = \frac{(-1)^n \lfloor n}{a\left(\frac{a}{\sin\theta}\right)^{n+1}} \left[\sin(n+1)\theta \right]$$

$$\Rightarrow \gamma_n = \frac{(-1)^n \lfloor n}{\frac{a^n \cdot a^{n+1}}{\sin^{(n+1)}\theta}} \left[\sin(n+1)\theta \right]$$

$$\Rightarrow \gamma_n = \frac{(-1)^n \lfloor n}{a^{(n+2)}} \left[\sin^{(n+1)}\theta \cdot \times \sin(n+1)\theta \right]$$ Ans

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Theorem

(1). If $y = \frac{1}{x^2+a^2}$, then prove that

$$\left[y_n = \frac{(-1)^n \lfloor n}{a^{(n+2)}} \sin^{(n+1)}\theta \times \sin(n+1)\theta \right]$$

Put $x = r\cos\theta$,

$$\boxed{r = \frac{x}{\cos\theta}}$$ in equ (ii)

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Maths

Q. Find y_n if $y = e^{ax} \sin bx$.

Sol^{n}:- Since, we have.

$$y = e^{ax} \sin bx.$$

$$\therefore y_1 = e^{ax} \cos bx \cdot b + \sin bx \cdot e^{ax} \cdot a.$$

$$\Rightarrow \boxed{y_1 = e^{ax}(a \sin bx + b \cos bx)} \quad \text{— (i)}$$

Put $a = r\cos\theta$ & $b = r\sin\theta$ and then squaring & adding (a & b), we get.

$$\boxed{a^2 + b^2 = r^2} \Rightarrow r = (a^2+b^2)^{1/2}$$

Now divide (b) by (a), we get

$$\frac{b}{a} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta.$$

$$\Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{b}{a}\right)}$$

Now, from eqn (i),

$$y_1 = e^{ax}[r\cos\theta \sin bx + r\sin\theta \cos bx].$$

$$\Rightarrow y_1 = re^{ax}[\sin bx \cos\theta + \cos bx \sin\theta]$$

$$\Rightarrow \boxed{y_1 = re^{ax}[\sin(bx+\theta)].} \quad [\because \sin A\cos B + \cos A \sin B = \sin(A+B)]$$

Similarly,

$$y_2 = r^2 e^{ax}[\sin(bx+2\theta)]$$

$$y_3 = r^3 e^{ax}[\sin(bx+3\theta)].$$

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$$y_n = r^n e^{ax}[\sin(bx+n\theta)]$$

$$\Rightarrow y_n = (a^2+b^2)^{n/2} e^{ax}\left[\sin\left(bx + n\tan^{-1}\left(\frac{b}{a}\right)\right)\right]$$ A.

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H.W 2: Find y_n if (i) $y = e^{ax}\cos bx$.

(ii) $y = e^{ax+b}\sin(cx+d)$.

(iii) $y = e^{ax+b}\cos(cx+d)$.

(iv) $y = e^{2x}\cos 3x$.

(v) $y = e^{2x+3}\cos(4x+5)$.

(vi) $y = e^{2x}\sin 3x$.

(vii) $y = e^{2x+1}\sin(3x+4)$.

(viii) $y = e^{x}\sin x$.

(ix) $y = e^{x}\cos x$.

Leibnitz Theorem.

State and prove Leibnitz Series (Theorem) if u & v are the functions of 'x' then the nth derivative of the product of u & v are

$$(uv)_n = {}^nC_0 u_n v_0 + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + {}^nC_3 u_{n-3} v_3 + \cdots + {}^nC_n u_0 v_n.$$

Proof: Let $y = uv$ ——— (i)

Diff. both sides successively, we get.

$$y_1 = u_1 v + u v_1$$

$$y_2 = (u_2 v + u_1 v_1) + (u_1 v_1 + u v_2)$$

$$y_2 = u_2 v + 2u_1 v_1 + u v_2.$$

$$y_3 = (u_3 v + u_2 v_1) + 2(u_2 v_1 + u_1 v_2) + (u_1 v_2 + u v_3)$$

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$\Rightarrow y_3 = u_3v + 3u_2v_1 + 3u_1v_2 + uv_3.$

$$\Rightarrow \boxed{y_3 = {}^3C_0 u_3 v_0 + {}^3C_1 u_{3-1} v_1 + {}^3C_2 u_{3-2} v_2 + {}^3C_3 u_0 v_3.}$$

$y_4 = (u_4v_0 + u_3v_1) + 3(u_3v_1 + u_2v_2) + 3(u_2v_2 + u_1v_3).$
$+ (u_1v_3 + uv_4).$

$\Rightarrow y_4 = u_4v_0 + 4u_3v_0 + 6u_2v_2 + 4u_1v_3 + u_0v_4$

$$\Rightarrow \boxed{y_4 = {}^4C_0 u_4 v_0 + {}^4C_1 u_3 v_1 + {}^4C_2 u_2 v_2 + {}^4C_3 u_1 v_3 + {}^4C_4 u_0 v_4}$$

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Thus, we see that the above process is true. Now, let us assumed that it will be true for $\boxed{n = m}$ (where m is any integers).

$$\boxed{y_m = (uv)_m = {}^mC_0 u_m v_0 + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + {}^mC_3 u_{m-3} v_3 + \cdots + {}^mC_m u_0 v_m.}$$

→ (ii)

Diff. both sides of eq(ii) w.r.t. x, we get.

$y_{m+1} = (uv)_{m+1} = {}^{m+1}C_0 (u_{m+1}v_0 + u_m v_1) + {}^{m+1}C_1 (u_m v_1 + {}^{m}C_2 u_{m-1} v_2)$

$+ {}^{m+1}C_3 u_{m-2} v_3 + \cdots \cdots$

$+ {}^mC_2 (u_{m-1} v_2 + u_{m-2} v_3) + \cdots \cdots$

$\cdots + {}^mC_m (u_1 v_m + u_0 v_{m+1}).$

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$$y_{m+1} = {}^{m}C_0 u_{m+1} v_0 + \left({}^{m}C_0 + {}^{m}C_1\right) u_m v_1 + \left({}^{m}C_1 + {}^{m}C_2\right) u_{m-1} v_2 + \left({}^{m}C_2 + {}^{m}C_3\right) u_{m-2} v_3 + \cdots + {}^{m}C_m u_0 v_{m+1} \quad \text{— (iii)}$$

We also know that by binomial theorem or binomial expansion.

$$\boxed{{}^{m}C_{r-1} + {}^{m}C_r = {}^{m+1}C_r} \quad [\text{where, } r = 0, 1, 2, 3, \ldots$$

$${}^{m}C_0 + {}^{m}C_1 = {}^{m+1}C_1$$

$${}^{m}C_1 + {}^{m}C_2 = {}^{m+1}C_2$$

$${}^{m}C_2 + {}^{m}C_3 = {}^{m+1}C_3$$

$$\Rightarrow \boxed{{}^{m}C_0 = {}^{m+1}C_0 = {}^{m}C_m = {}^{m+1}C_{m+1} = 1} \quad \text{— (iv)}$$

Putting the value of equ (iv) in equ (iii), we get

$$y_{m+1} = {}^{m+1}C_0 u_{m+1} v_0 + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1} v_2 + {}^{m+1}C_3 u_{m-2} v_3 + \cdots + {}^{m+1}C_{(m+1)} u_0 v_{m+1}$$

$$y_{m+1} = {}^{m+1}C_0 u_{m+1} v_0 + {}^{m+1}C_1 u_{(m+1)-1} v_1 + {}^{m+1}C_2 u_{(m+1)-2} v_2 + {}^{m+1}C_3 u_{(m+1)-3} v_3 + \cdots + {}^{m+1}C_r u_{(m+1)-r} v_r + \cdots + {}^{m+1}C_{m+1} u_{(m+1)-(m+1)} v_{m+1}$$

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Hence, we see that the above process is proved for $m = 1, 2, 3, 4, \cdots$.

Therefore, we see that our process is true for $m = 1, 2, 3, 4, \cdots, n, (n+1), (n+2), \cdots$.

Thus, we see that our process is true for all integral values of m.

So, we conclude that,

$$v_m = {}^nC_0 u_n v_0 + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_r u_{n-r} v_r + \cdots + {}^nC_n u_0 v_n.$$

i.e., $(v)_m = (uv)_m$.

$$(uv)_m = {}^nC_0 u_n v_0 + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_r u_{n-r} v_r + \cdots + {}^nC_n u_0 v_n.$$

Hence proved

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Q: If $y = e^{m \sin^{-1} x}$, then prove that:

(I) $\boxed{(1-x^2) y_2 - x y_1 - m^2 y = 0}$.

(II) $\boxed{(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (m^2+n^2) y_n = 0}$

⇒ Proof:- (I) We have, $y = e^{m \sin^{-1} x}$ — (i)

Diff. both sides w.r.t. x, we get.

$$y_1 = e^{m \sin^{-1} x} \left(\frac{m}{\sqrt{1-x^2}} \right).$$

$$\Rightarrow \left(\sqrt{1-x^2}\right) y_1 = m e^{m \sin^{-1} x}.$$

Using equ (i), the above equ is written as.

$$\boxed{\left(\sqrt{1-x^2}\right) y_1 = m y}$$

Squaring both sides, we get

$$\boxed{(1-x^2) y_1^2 = m^2 y^2}$$

Again diff. both sides. w.r.t. x, we get.

$$2 y_1 y_2 (1-x^2) + y_1^2 (-2x) = 2 m^2 y y_1$$

$$\Rightarrow 2 y_1 \left[(1-x^2) y_2 - 2x y_1 \right] = 2 y_1 \left[m^2 y \right]$$

$$\Rightarrow \left[(1-x^2) y_2 - 2x y_1 \right] = m^2 y$$

$$\Rightarrow \boxed{(1-x^2) y_2 - x y_1 - m^2 y = 0} \quad \text{(I)}$$

proved

For II

The above soln (I) is written as.

$$y_2(1-x^2) - xy_1 = m^2 y$$

Diff. both sides w.r.t. x n times by Leibnitz theorem term by term, we get.

$$[y_2(1-x^2)]_n - [y_1 x]_n - [m^2 y]_n = 0$$

By using the formulae, we get.

$${}^nC_0 y_{n+2}(1-x^2) + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2) + \text{(all terms be vanished (=0))}$$

$$- [{}^nC_0 y_{n+1} x + {}^nC_1 y_n x' + \text{all terms be zero}]$$

$$- m^2 y_n = 0.$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nx y_{n+1} + -2\cdot\frac{n(n-1)}{2!} y_n + 0.$$

$$- x y_{n+1} - n y_n + 0 - m^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nx y_{n+1} - n^2 y_n + n y_n - x y_{n+1} - n y_n$$

$$- m^2 y_n = 0$$

$$\Rightarrow \boxed{(1-x^2)y_{n+2} - (2n+1)x y_{n+1} - (m^2+n^2)y_n = 0}$$

Hence proved

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Maclaurin's Series (Theorem)

Expansion of functions

State and proof Maclaurin's Series or Theorem.

Statement :- If $f(x)$ be a fn of 'x', which can be expand in powers of 'x' and let the expansion be differential term by term, any no. of times, then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^{n}(0)$$

⇒ **Proof :-** let $f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + A_5 x^5 + \cdots$ —— (i)

[where, $A_0, A_1, A_2, A_3, A_4 \cdots$ are constants to be determined]

Diff. equ (i) successively w.r.t. x, we get

$$f'(x) = 0 + A_1 + 2A_2 x + 3A_3 x^2 + 4A_4 x^3 + \cdots \quad \text{—(ii)}$$

$$f''(x) = 0 + 0 + 2A_2 + 6A_3 x + 12A_4 x^2 + 20A_5 x^3 + \cdots \quad \text{—(iii)}$$

$$f'''(x) = 0 + 6A_3 + 24A_4 x + 60A_5 x^2 + \cdots \quad \text{(iv)}$$

$$f^{iv}(x) = 0 + 24A_4 + 120A_5 x + \cdots \quad \text{(v)}$$

$f^{v}(x)$ — — —

J.M.S

$A_0 = f(0)$.

$A_1 = f'(0)$.

$A_2 = \frac{1}{\lfloor 2} f''(0)$.

$A_3 = \frac{1}{\lfloor 3} f'''(0)$

$A_4 = \frac{1}{\lfloor 4} f^{iv}(0)$ ....

Substituting these values in above equ (i)

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

hence proved

Note !!

(i) If $y = f(x)$, $y_0 = f(0)$; $(y_1)_0 = f'(0)$, $(y_2)_0 = f''(0)$, $(y_3)_0 = f'''(0)$, ---- ,

then Maclaurin's series takes the form

$$y = y_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \cdots$$

8 ⇒ Expand $\sin x$ by Maclaurin's series

Soln: Let $y = \sin x$.

$y_1 = \cos x$.

$y_2 = -\sin x$.

$y_3 = -\cos x$

$y_4 = +\sin x$.

Putting $x = 0$, we get

$y = 0$.

$y_1 = 1$

$y_2 = 0$.

$y_3 = -1$

$y_4 = 1$

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Putting $x=0$.

Now, from Maclaurin's series.

$$y = y_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \cdots$$

Therefore,

$$\sin x = 0 + x \cdot 1 + 0 - \frac{x^3}{3!} \cdot 1 + \cdots$$

$$\Rightarrow \boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

Ans

2nd Method

Let $\boxed{y = \sin x}$.

$\therefore \boxed{y_n = \sin\left(\frac{n\pi}{2} + x\right)}$

Put $x=0$, then

$\boxed{(y_n)_0 = \sin\frac{n\pi}{2}}$,

Again put $n = 0, 1, 2, 3, \ldots$ in $(y_n)_0$, we get

$(y)_0 = \sin\frac{0\pi}{2} = 0$.

$(y_1)_0 = \sin\frac{\pi}{2} = 1$.

$(y_2)_0 = \sin\pi = 0$.

$(y_3)_0 = \sin\frac{3\pi}{2} = -1$.

$(y_4)_0 = \sin 2\pi = 0$.

Now, from Maclaurin's series,

$$y = y_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \cdots$$

Therefore,

$$\sin x = 0 + x \cdot 1 + 0 - \frac{x^3}{3!} + \cdots$$

$\sin x =$

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Taylor's Theorem or Series

State & prove Taylor's Theorem.

Statement :- Let $f(x+h)$ be a function of h which can be expanded in powers of x and let the expansion be differential any no. of times w.r.t. 'h', then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots + \frac{h^n}{n!}f^n(x) + \cdots$$

⇒ **Proof :-** Let $f(x+h) = A_0 + A_1h + A_2h^2 + A_3h^3 + A_4h^4 + A_5h^5 + \cdots$ — (i)

[where, $A_0, A_1, A_2, A_3 \cdots$ are constants to be determined]

Diff. equ (i) successively w.r.t. 'h', we get

$$f'(x+h) = 0 + A_1 \cdot 1 + 2A_2h + 3A_3h^2 + 4A_4h^3 + 5A_5h^4 + \cdots$$ — (ii)

$$f''(x+h) = 0 + 2 \cdot 1 A_2 + 3.2.A_3h + 4.3A_4h^2 + 5.4.A_5h^3 + \cdots$$

$$f'''(x+h) = 0 + 3.2A_3 + 4.3.2A_4h + 5.4.3.A_5h^2 + \cdots$$ — (iii)

— (iv)

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$f^{iv}(x+h) = 0 + 4.3.2.A_4 + 5.4.3.A_5 h + \cdots$ ↳ (v)

Putting $x = 0$ in above equns (i), (ii), (iii), (iv), (v)

$A_0 = f(x)$.

$A_1 = f'(x)$.

$A_2 = \frac{f''(x)}{2!}$

$A_3 = \frac{f'''(x)}{3!}$

$A_4 = \frac{f^{iv}(x)}{4!}$

Substituting these values in equn (i), we get

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x) + \cdots + \frac{h^n}{n!}f^n(x) + \cdots$$

hence proved

Q. Show that $\log(x+h) = \log h$.

1.8 Show that $\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} + \cdots$

⇒ Sol^n:- Let $y = lx$ — (A)

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Note!!

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$
$$\cdots + \frac{h^n}{n!}f^n(x) + \cdots$$

Note (I) Putting $\boxed{x = a}$

(i) $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots$
$$\cdots + \frac{h^n}{n!}f^n(a) + \cdots \rightarrow (A)$$

Note!! (ii):- Putting $\boxed{x = h \text{ & } h = a}$ in Taylor's series, we have

$$f(a+h) = f(h) + af'(h) + \frac{a^2}{2!}f''(h) + \frac{a^3}{3!}f'''(h)$$
$$+ \cdots + \frac{a^n}{n!}f^n(h) + \cdots$$

Note (iii) :- Putting $\boxed{h = x-a}$ in above Note (i)

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a)$$
$$+ \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Note (iv) Putting $\boxed{h = x \text{ & } x = h}$ in Taylor's series,

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h)$$
$$+ \frac{x^4}{4!}f^{iv}(h) + \cdots + \frac{x^n}{n!}f^n(h) + \cdots$$

(B)

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8. Show that $\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} + \cdots$

⇒ Soln:- Proof:- Here we are to expand in powers of x.

Thus, we are to use the form

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \cdots + \frac{x^n}{n!}f^n(h) + \cdots \quad \text{(i)}$$

We have,

$f(x+h) = \log(x+h)$.

Putting $x = 0$ in above equ, we get.

$$f(h) = \log h.$$

$$f'(h) = \frac{1}{h}$$

$$f''(h) = -\frac{1}{h^2} = -h^{-2}$$

$$f'''(h) = +\frac{2}{h^3} = 2h^{-3} = \frac{2}{h^3}$$

$f^{iv}(h) = \cdots$

Hence, from equ (i)

$$f(x+h) = f(h) + x\frac{1}{h} + \frac{x^2}{2!}\left(-\frac{1}{h^2}\right) + \frac{x^3}{3!}\frac{2}{h^3} + \cdots$$

$$\Rightarrow f(x+h) = f(h) + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} + \cdots$$

proved

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(2) Expand $\sin^{-1}(x+h)$ in powers of x as far as term in x^3.

Soln:- We have,

$$f(x+h) = \sin^{-1}(x+h) \quad \text{— (i)}$$

Putting $x=0$, we get.

$$f(h) = \sin^{-1}(h)$$

$$\therefore f'(h) = \frac{1}{\sqrt{1-h^2}} = \frac{1}{(1-h^2)^{1/2}} = (1-h^2)^{-1/2}.$$

$$\therefore f''(h) = -\frac{1}{2}(1-h^2)^{-\frac{3}{2}} \times (-2h) = -\frac{1}{2} \times \frac{1(-2h)}{(1-h^2)^{3/2}}.$$

$$\& \quad f'''(h) = +\frac{1}{2} \times \frac{3}{2}(1-h^2)^{-5/2} = \frac{3}{4} \times \frac{1}{(1-h^2)^{5/2}}.$$

$$f''''(h) = +\frac{2h}{2(1-h^2)^{3/2}} = h(1-h^2)^{-3/2}$$

$$f'''(h) = h\left\{-\frac{3}{2}(1-h^2)^{-5/2} \times (-2h)\right\} + (1-h^2)^{-3/2} \cdot 1.$$

$$\Rightarrow f'''(h) = 3h^2(1-h^2)^{-5/2} + (1-h^2)^{-3/2}.$$

Putting these values in (B), we get

$$\sin^{-1}(x+h) = \sin^{-1}(h) + x(1-h^2)^{-1/2} + \frac{x^2}{2!}h(1-h^2)^{-3/2} + \cdots$$

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Q. Show that $\sin(x+h) = \sin x + h\cos x - \frac{h^2}{2!}\sin x - \frac{h^3}{3!}\cos x + \dots$

⇒ Proof :- We have,

$$f(x+h) = \sin(x+h).$$

Putting $h=0$, we get

$$f(x) = \sin x.$$

$$\therefore f'(x) = \cos x.$$

$$f''(x) = -\sin x.$$

$$f'''(x) = -\cos x$$

Putting these values in (A), we get.

~~$f(x+h) = f$~~

$$\Rightarrow \sin(x+h) = \sin x + h\cos x + \frac{h^2}{2!}(-\sin x) + \frac{h^3}{3!}(-\cos x) + \dots$$

VVI:

~~Proof of th.~~

Prove that $\tan^{-1}(x+h) = \tan^{-1}x + h\sin z \cdot \frac{\sin z}{1} - \frac{(h\sin z)^2}{2}\sin 2z + (h\sin z)^3 \frac{\sin}{3}$ + - - - - - - - - - -

⇒ **Proof:-** where $z = \cot^{-1}x$

⇒ **Proof:-** Let $f(x+h) = \tan^{-1}(x+h)$. | $\because z = \cot^{-1}x$.

Put $h = 0$ both sides, then | $\Rightarrow x = \cot z$.

$f(x) = \tan^{-1}x$ — (i) | $\therefore \frac{dx}{dz} = -\cos$

Diff. equ (i) w.r.t. x, we get.

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1+\cot^2 z} = \frac{1}{\operatorname{cosec}^2 z} = \sin^2 z.$$

$$\therefore f''(x) = 2\sin z \cdot \cos z \cdot \frac{dz}{dx} = \sin 2z \cdot \frac{dz}{dx} \quad [\because 2\sin\theta\cos\theta = \sin 2\theta]$$

$$f'''(x) = 2\left(\sin z(-\sin z) + \cos z \cdot \cos z\right)\frac{dz}{dx}$$

$$f'''(x) = \left(-2\sin^2 z + 2\cos^2 z\right)\cdot(-\sin^2 z)$$

$$\Rightarrow f'''(x) = -4\sin^4 z$$

$$\therefore f''(x) = 2\cos 2z \cdot \frac{dz}{dx}$$

$$\Rightarrow f''(x) = \sin 2z \cdot (-\sin^2 z)$$

$$\Rightarrow f''(x) = \left[(-\sin^2 z)\cos 2z \cdot 2\frac{dz}{dx} + \sin 2z \cdot (-1)2\sin z \cdot \cos z\right.$$

$$\Rightarrow f'''(x) = \left[-2\sin^2 z \cdot \cos 2z - 2\sin 2z \cdot \sin z \cdot \cos z\right)$$

$$= -\left[2\sin^2 z \cdot \cos 2z + 2\sin$$

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$= \sin^2 z \left[2\sin^2 z \cdot \cos^2 z + 2 \times 2 \sin z \cos z \cdot \sin z \cdot \cos z \right]$

$= 2\sin^4 z \left[\cos^2 z + 2\cos^2 z \right]$.

$= 2\sin^4 z \times 3\cos^2 z \left[\cos^2 z - \sin^2 z + 2\cos^2 z \right]$

$= 2\sin^4 z \left[3\cos^2 z - \sin^2 z \right] + 2(1 - \sin^2 z)$.

$= 2\sin^4 z \left[\cos^2 z + 2 - 2\sin^2 z \right]$

$= 2\sin^4 z \left[\cos^2 z - 3\sin^2 z + 2 \right]$.

$= 2\sin^4 z \left[1 - \sin^2 z - 3\sin^2 z + 2 \right]$.

$= 2\sin^4 z \left[3 - 4\sin^2 z \right]$.

$= 2\sin^3 z \left[3\sin z - 4\sin^3 z \right]$.

$f'''(x) = 2\sin^3 z \times \sin 3z$ $\left[\because \sin 3\theta = 3\sin\theta - 4\sin^3\theta \right]$

Now, by Taylor's theorem, we have,

~~$f(x+h) = \tan$~~

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$

$\therefore \tan^{-1}(x+h) = \tan^{-1} x + h \cdot \sin^2 z - \frac{h^2}{2!} \sin 2z \cdot \sin^2 z + \frac{h^3}{3!} 2\sin^3 z \times \sin 3z + \dots$

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1. Q. Expand $\log \sin(x+h)$ in powers of h by Taylor's Theorem.

2. Q. Expand $\log \cos(x+h)$ in powers of h by Taylor's Theorem.

3. Q. Express the polynomial $2x^3+7x^2+x-1$ in powers of $(x-2)$.

⇒ Soln:- Here, we have,

$$f(x) = 2x^3+7x^2+x-1 \quad \text{---(i)}$$

$$\therefore f'(x) = 6x^2+14x+1 \quad \text{---(ii)}$$

$$f''(x) = 12x+14 \quad \text{---(iii)}$$

$$f'''(x) = 12. \quad \text{---(iv)}$$

$$f^{iv}(x) = 0. \quad \text{---(v)}$$

Here, we are to use the following formula

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Putting $a=2$ in the above formula,

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \cdots$$

Putting $x=2$ in above eqns (i), (ii), (iii), (iv) & (v)

$f(2) = 16 + 28 + 2 - 1 = 45$

$f'(2) = 24 + 28 + 1 = 53.$

$f''(2) = 24 + 14 = 38.$

$f'''(2) = 12 = 12$

$f^{iv}(2) = 0 = 0$

Putting these values in equ (vi), we get.

$f(x) = 45 + (x-2)53 + \frac{(x-2)^2}{2!}38 + \frac{(x-2)^3}{3!}12 + \cdots$

$\Rightarrow \boxed{2x^3 + 7x^2 + x - 1 = 45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3.}$

Ans

4. Q. Expand e^x in powers of $(x-1)$

5. Q. Expand $\sin x$ in powers of $(x - \frac{\pi}{2})$.

6. Q. Expand $\log \sin x$ in powers of $(x-a)$.

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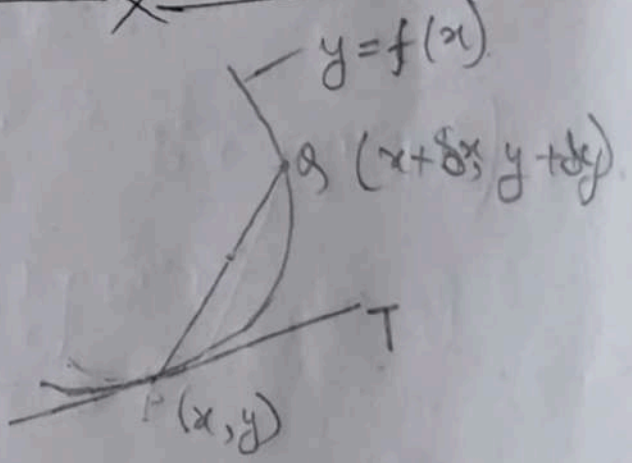
. Tangents & Normals.

Tangents & Normals.



Let P & Q be two neighbouring points as Q ~~is~~ tends to P (i.e. approaches P) along the curve, the straight line joining P & Q tends to a definite straight line PT, called the tangent to the curve to P.

Q. To find the equ of tangent to the curve at any point of it.

⇒ Soln:- Let P(x,y) be the given point of the curve.

$y = f(x)$.

Let Q$(x+\delta x, y+\delta y)$ be a point neighbouring to P on the curve. Then the equ of PQ is:-

$$Y - y = \frac{(y+\delta y)-y}{(x+\delta x)-x}(X-x).$$

$$\Rightarrow \boxed{(Y-y) = \frac{dy}{dx}(X-x)}$$

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Now, as Q tends to P.

Therefore, $\delta y \to 0$, $\delta x \to 0$.

$\frac{\delta y}{\delta x} \to \frac{dy}{dx}$ & the line PQ tends to tangent line PT. to the

$PQ \to PT$

curve at P. So, the equo of tangent to the curve $y = f(x)$ & $P(x, y)$ is $(Y - y) = \frac{dy}{dx}(X - x)$

Q. Find the equo of the tangent at (x, y) to the curve $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$.

⇒ **Soln:-** The given curve is $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$ ——— (i)

Diff. both sides w.r.t. x, we get

$$\frac{mx^{m-1}}{a^m} + \frac{my^{m-1}}{b^m}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\frac{mx^{m-1}}{a^m}}{\frac{my^{m-1}}{b^m}} = -\frac{b^m \cancel{m} x^{m-1}}{a^m \cdot \cancel{m} y^{m-1}}$$

Now, the equo of the tangent is

$$(Y - y) = \frac{dy}{dx}(X - x) \Rightarrow (Y - y) = -\left(\frac{b^m x^{m-1}}{a^m y^{m-1}}\right) \times (X - x)$$

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$\Rightarrow (y-y) =$

$\Rightarrow a^m (y-y) y^{m-1} = -b^m (x-x) x^{m-1}.$

$\Rightarrow \boxed{a^m y y^{m-1} - a^m y^m = -b^m x x^{m-1} + b^m x^m.}$

$\Rightarrow \left[a^m y y^{m-1} + b^m x x^{m-1} = b^m x^m + a^m y^m. \right] \div a^m b^m.$

Dividing both sides by $(a^m b^m)$, we get

$\Rightarrow \frac{y y^{m-1}}{b^m} + \frac{x x^{m-1}}{a^m} = \frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$

$\Rightarrow \boxed{\frac{x x^{m-1}}{a^m} + \frac{y y^{m-1}}{b^m} = \frac{x^m}{a^m} + \frac{y^m}{b^m}}$

$\left[\because \frac{x^m}{a^m} + \frac{y^m}{b^m} = 1 \right]$ from equ (i)

Hence, the required equ of the tangent is $\boxed{\frac{x x^{m-1}}{a^m} + \frac{y y^{m-1}}{b^m} = 1}$ proved

H.W.

1. Find the equ of the tangent at (x_1, y_1) to the ellipse $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$.

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Asymtotes (অনন্তমুখী)

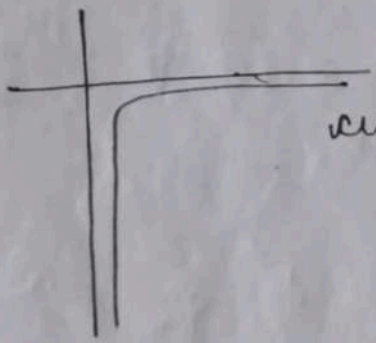
Definition :- A straight line at a infinite distance from the origin to which a tangent to a curve tends, as the point of contact tends to infinity, is called an asymtotes of the curve.

In other words,

A straight line which cuts the curve at infinity but does not lie absolutely at infinity is called an asymtotes of the curve.

B

Q. Find all the asymtotes of the curve $x^3 + 2x^2y - xy^2 - 2y^3 + x^2 - y^2 - 2x - 3y = 0$.



⇒ **Solⁿ:-** The given curve is

$$(x^3 + 2x^2y - xy^2 - 2y^3) + (x^2 - y^2) - (2x + 3y) = 0 \quad \text{---(i)}$$

Working rule 1: Putting $x = 1$ & $y = m$ in the highest degree terms in (i), we get $\{\phi_n(m)\}$

$$\phi_n(m) = \phi_3(m) = 1 + 2m - m^2 - 2m^3$$

$$\left[\because \phi_3(x, y) = x^3 + 2x^2y - xy^2 - 2y^3\right]$$

$$\Rightarrow \phi_n'(m) = \phi_3'(m) = 0 + 2 - 2m - 6m^2$$

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Working rule (ii) :- Equating $\phi_n(m) = \phi_3(m) = 0$.

$\phi_n(m) = 0 \Rightarrow 1 + 2m - m^2 - 2m^3 = 0$

$\Rightarrow (1+2m) - m^2(1+2m) = 0$

$\Rightarrow (1+2m)(1-m^2) = 0$.

$\Rightarrow (1+2m)(1+m)(1-m) = 0$.

$\Rightarrow \boxed{m = -\frac{1}{2}}$ or $\boxed{m = -1}$ or $\boxed{m = 1}$.

i.e., $\boxed{m_1 = 1}$, $\boxed{m_2 = -1}$, $\boxed{m_3 = -\frac{1}{2}}$

Working rule (iii) :- Also putting $\boxed{x = 1}$, $\boxed{y = m}$ in the 2nd highest degree term:

$\boxed{[\phi_{(n-1)}(m) = \phi_{3-1}(m) = (1-m^2)] = \phi_2(m)}$

~~Wkt. Working rule (iv)~~ We also know that $c = -\frac{\phi_{(n-1)}(m)}{\phi_n'(m)} = -\frac{\phi_2(m)}{\phi_3'(m)}$

So, $\boxed{c = -\frac{1-m^2}{2-2m+6m^2}}$

Working rule (v) :- When $m = 1 = m_1$ (say),

$c = 0 = c_1$ (say).

~~Put m~~ When $m = -1$ $(= m_2)$ (say)

$\therefore c = 0 = c_2$ (say).

And when $m = -\frac{1}{2} = m_3$ (say)

$\therefore c = -\frac{1-(-\frac{1}{2})^2}{2-2\times(-\frac{1}{2})-6\times(-\frac{1}{2})^2} = -\frac{1-\frac{1}{4}}{2+1-\frac{6}{4}} = -\frac{\frac{3}{4}}{3-\frac{6}{4}}$

$= -\frac{\frac{3}{4}}{\frac{3}{2}} = -\frac{1}{2}$

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Hence, the required asymtotes are :—

$y_a = mx + c$

$\Rightarrow y_1 = m_1 x + c_1 \Rightarrow y_1 = 1.x + 0 \Rightarrow \boxed{x - y = 0}$

$y_2 = m_2 x + c_2 \Rightarrow y = -1.x + 0 \Rightarrow \boxed{x + y = 0}$

$y_3 = m_3 x + c_3 \Rightarrow y_3 = -\frac{1}{2}x - \frac{1}{2} = -\frac{1}{2}(x+1) = 0$

$\Rightarrow 2y \Rightarrow \boxed{x + 2y + 1 = 0}$

W.J

1.8 Find the asymtotes of the curve $x^3 + y^3 - 3axy = 0$ Ans

$\Rightarrow$ Soln:— $\phi_3(m) = 1 + m^3$ | $\phi_2(m) = -3am$.

$\phi_3'(m) = 3m^2$

Equating $\phi_3(m) = 0$, ie, $\boxed{\phi_3(m) = 0}$

$\Rightarrow 1 + m^3 = 0$

$\Rightarrow (1+m)(1+m^2-m) = 0.$

$\Rightarrow \boxed{m = -1}$, $m^2 - m + 1 = 0$

$\Rightarrow m^2 - 2m + m + 1 = 0$

For $m^2 - m + 1 = 0$,

Here $a = 1$, $b = -1$, $c = 1$.

$\therefore \alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2 \times 1}$

$\Rightarrow \alpha, \beta = \frac{-1 \pm \sqrt{-3}}{2}$

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$\alpha, \beta = \frac{-1 \pm \sqrt{3i^2}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$

$\Rightarrow \boxed{\alpha = \frac{-1+\sqrt{3}i}{2}}, \boxed{\beta = \frac{-1-\sqrt{3}i}{2}}$

Hence, the required asymptotes are

$\because \boxed{y = mx+c}$ $\quad c = -\frac{(-3am)}{3m^2} = \frac{a}{m}$

$\Rightarrow c = \frac{a}{m}$

When $m = -1$, $\Rightarrow \boxed{c = -a}$

When $m = \frac{1+\sqrt{3}i}{2}$, then $c = \frac{a}{\frac{1+\sqrt{3}i}{2}}$

$\Rightarrow \boxed{c = \frac{2a}{1+\sqrt{3}i}}$

When $m = \frac{1-\sqrt{3}i}{2}$, then $\boxed{c = \frac{2a}{1-\sqrt{3}i}}$

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(2) Q. Find all the asymptotes of the following:

(i) $y^3 - 3x^2y + xy^2 - 3x^2 + 2y^2 + 2xy + 9x + 5y + 16 = 0$

(ii) $4x^3 - 8x^2y - 4xy^2 + y^3 + 3x^2 + 2xy - y^2 - 7 = 0$

(iii) $x^3 - 4xy^2 - 3x^2 + 12xy - 12y^2 + 8x + 2y + 4 = 0$

(iv) $2x^3 + 3x^2y - 3xy^2 - 2y^3 + 3x^2 - 3y^2 + y - 3 = 0$

(v) $y^3 - x^2y - 2xy^2 + 2x^3 - 7xy - 3y^2 + 12x^2 + 2x + 2y + 1 = 0$

(vi) $3x^3 + 2x^2y + 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0$

(vii) $y^3 - 2y^2x - yx^2 + 2x^3 + x^2 - 6xy + 5y^2 - 2y + 2x + 1 = 0$

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VVI

Tangent & Normal.

Q) If $p = x\cos\alpha + y\sin\alpha$ touches the curve $\left(\frac{x}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y}{b}\right)^{\frac{n}{n-1}} = 1$., prove that $p^n = (a\cos\alpha)^n + (b\sin\alpha)^n$.

⇒ **Proof:-** Let the line $(p = x\cos\alpha + y\sin\alpha)$ → (i) touches the curve $\left(\frac{x}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y}{b}\right)^{\frac{n}{n-1}} = 1$. → (ii) at the point (x_1, y_1).

Differ. equ (ii) w.r.t. x, we get

$$\frac{n}{n-1}\left(\frac{x}{a}\right)^{\left[\frac{n}{n-1}-1\right]} \times \frac{1}{a} + \frac{n}{n-1}\left(\frac{y}{b}\right)^{\frac{n}{n-1}-1} \times \frac{1}{b}\frac{dy}{dx} = 0,$$

$$\Rightarrow \frac{n}{n-1}\left(\frac{x}{a}\right)^{\left(\frac{1}{n-1}\right)} \times \frac{1}{a} + \frac{n}{n-1}\left(\frac{y}{b}\right)^{\frac{1}{n-1}} \times \frac{1}{b}\frac{dy}{dx} = 0.$$

$$\Rightarrow \frac{n}{n-1}\left(\frac{x}{a}\right)^{\frac{1}{n-1}} \cdot \frac{1}{a} = -\frac{n}{n-1}\left(\frac{y}{b}\right)^{\frac{1}{n-1}} \times \frac{1}{b}\frac{dy}{dx}$$

V.V.I

$$\Rightarrow \frac{1}{a}\left(\frac{x}{a}\right)^{\frac{1}{n-1}} = -\frac{1}{b}\left(\frac{y}{b}\right)^{\frac{1}{n-1}} \cdot \frac{dy}{dx}.$$

$$\Rightarrow \frac{dy}{dx} = -\frac{b}{a}\left(\frac{x \times b}{a \times y}\right)^{\frac{1}{n-1}}$$

At the point (x_1, y_1), $\boxed{\frac{dy}{dx} = -\frac{b}{a}\left(\frac{bx_1}{ay_1}\right)^{\frac{1}{n-1}}}$ → (iii)

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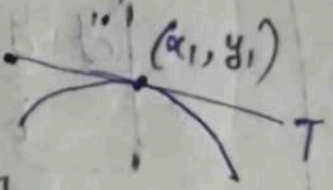
The equo of tangent at the point (x_1, y_1)

$$\Rightarrow y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$



$$\Rightarrow \boxed{(y - y_1) = -\frac{b}{a}\left(\frac{bx_1}{ay_1}\right)^{\frac{1}{n-1}} \cdot (x - x_1)}$$

$$\Rightarrow (y - y_1)\, a \cdot (ay_1)^{\frac{1}{n-1}} = -b \cdot (bx_1)^{\frac{1}{n-1}} (x - x_1)$$

$$\Rightarrow (y - y_1)\, a^{\frac{n}{n-1}} \cdot y_1^{\frac{1}{n-1}} = -b^{\frac{n}{n-1}} \cdot x_1^{\frac{1}{n-1}} (x - x_1)$$

$$\Rightarrow y a^{\frac{n}{n-1}} \cdot y_1^{\frac{1}{n-1}} - y_1 a^{\frac{n}{n-1}} \cdot y_1^{\frac{n}{n-1}} = -b^{\frac{n}{n-1}} \cdot x_1 \cdot x_1^{\frac{1}{n-1}} + b^{\frac{n}{n-1}} \cdot x_1^{\frac{n}{n-1}}$$

$$\Rightarrow \boxed{y a^{\frac{n}{n-1}} \cdot y_1^{\frac{1}{n-1}} + x b^{\frac{n}{n-1}} \cdot x_1^{\frac{1}{n-1}} = b^{\frac{n}{n-1}} \cdot x_1^{\frac{n}{n-1}} + a^{\frac{n}{n-1}} \cdot y_1^{\frac{n}{n-1}}}$$

Dividing the above equo by $a^{\frac{n}{n-1}} \cdot b^{\frac{n}{n-1}}$.

$$\Rightarrow \boxed{\frac{y \cdot y_1^{\frac{1}{n-1}}}{b^{\frac{n}{n-1}}} + \frac{x \cdot x_1^{\frac{1}{n-1}}}{a^{\frac{n}{n-1}}} = \frac{x_1^{\frac{n}{n-1}}}{a^{\frac{n}{n-1}}} + \frac{y_1^{\frac{n}{n-1}}}{b^{\frac{n}{n-1}}}}$$

$$\frac{y \cdot y_1^{\frac{1}{n-1}}}{b^{\frac{n}{n-1}}} + \frac{x x_1^{\frac{1}{n-1}}}{a^{\frac{n}{n-1}}} = \left(\frac{x_1}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y_1}{b}\right)^{\frac{n}{n-1}}$$

$\because \left(\frac{x_1}{a}\right)^{\frac{n}{n-1}}$ From equo (ii),

$$\boxed{\left(\frac{x_1}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y_1}{b}\right)^{\frac{n}{n-1}} = 1} \rightarrow (ii)(a)$$

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Using (2a),

$$\frac{x \cdot x_1^{\frac{1}{n-1}}}{a^{\frac{n}{n-1}}} + \frac{y \cdot y_1^{\frac{1}{n-1}}}{b^{\frac{n}{n-1}}} = 1 \quad \text{— (iv)}$$

If line 1 touches the curve (2) at (x_1, y_1) then line 1 & line 4 represents the same line.

Hence, comparing equ (i) & (iv)

$$\frac{\cos\alpha}{\frac{x_1^{\frac{1}{n-1}}}{a^{\frac{n}{n-1}}}} = \frac{\sin\alpha}{\frac{y_1^{\frac{1}{n-1}}}{b^{\frac{n}{n-1}}}} = \frac{p}{1}$$

$\Rightarrow$ Therefore, $\cos\alpha = p\left(\frac{x_1^{\frac{1}{n-1}}}{a^{\frac{n}{n-1}}}\right) \times a$

$\Rightarrow a\cos\alpha = p\frac{x_1^{\frac{1}{n-1}}}{a^{\frac{n}{n-1}}} \cdot a$

$\Rightarrow a\cos\alpha = p x_1^{\frac{1}{n-1}} \cdot a^{\frac{1}{n-1}}$

$\Rightarrow \boxed{a\cos\alpha = p\left(\frac{x_1}{a}\right)^{\frac{1}{n-1}}}$ — (v)

$\sin\alpha = p\left(\frac{y_1^{\frac{1}{n-1}}}{b^{\frac{n}{n-1}}}\right) \times b$

$\Rightarrow \boxed{b\sin\alpha = p\left(\frac{y_1}{b}\right)^{\frac{1}{n-1}}}$ — (vi)

Rising equ (v) & (vi) to power n, and adding

$$\Rightarrow (a\cos\alpha)^n + (b\sin\alpha)^n = p^n\left[\left(\frac{x_1}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y_1}{b}\right)^{\frac{n}{n-1}}\right] \quad \text{— (vii)}$$

~~From eq~~ ~~for~~ From equ (ii) a & (vii),

$$(a\cos\alpha)^n + (b\sin\alpha)^n = p^n.$$ proved

VVI.

(2) Prove that the condition that $x\cos\alpha + y\sin\alpha = p$ touches the curve $x^m y^n = a^{m+n}$ is

$$p^{m+n}\, m^m . n^n = (m+n)^{m+n} \cdot a^{m+n} \cdot \begin{matrix}\cos^m\alpha \\ \sin^n\alpha\end{matrix}$$

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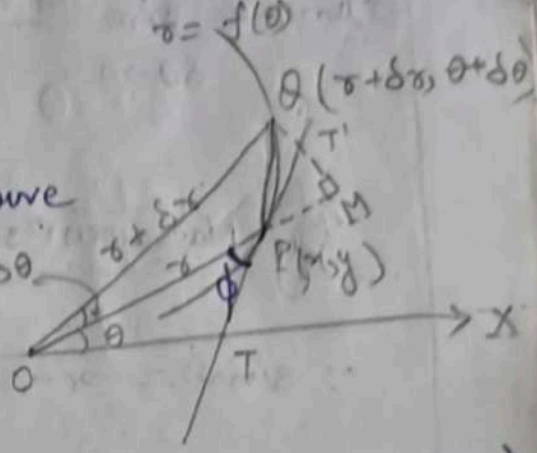
Q Find the angle b/w tangent & radius vector.

Soln:- Let T be two neighbouring points on the curve $r = f(\theta)$ is the pole & Ox, $\delta\theta$ is the initial line join ~~O→P~~ O–P, O–Q. & Q → P.



From QM, QM ⊥ OP (proceed as necessary).

T–P–T' is the tangent of the curve at P.

Let ϕ be the angle b/w the tangents P–T & the radius vector OP.

Now, as Q → P, $\delta r \to 0$, $\delta\theta \to 0$ line OQ → line OP

Scant PQ → tangent PT

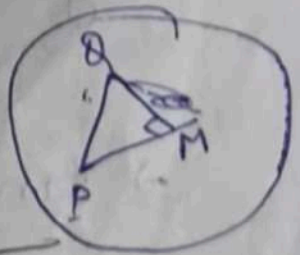
& ∠QPM → ϕ

Therefore, $\tan\phi = \tan \lim_{\delta\theta\to 0} \angle QPM = \lim_{\delta\theta\to 0} \tan\angle QPM$

$= \lim_{\delta\theta\to 0} \frac{QM}{PM}$.

In right angled Δ,

$\tan QPM = \frac{QM}{PM}$



$\therefore \tan\phi = \lim_{\delta\theta\to 0} \frac{(r+\delta r)\sin\delta\theta}{OM - OP}$

$= \lim_{\delta\theta\to 0} Q'$

[$\because$ In Δ OMQ, $\frac{\sin\delta\theta}{QM} = \frac{}{r+\delta r}$ ⇒ $(r+\delta r)\sin\delta\theta = QM$

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$$\Rightarrow \tan\phi = \lim_{\delta\theta \to 0} \frac{(r+\delta r)\sin\delta\theta}{(r+\delta r)\cos\delta\theta - r} \quad [\because OP = r]$$

[$\because$ In $\triangle OMQ$, $\cos\delta\theta = \frac{OM}{r+\delta r}$]

By Maclaurin's Series,

$$\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \cdots \Rightarrow \boxed{(r+\delta r)\cos\delta\theta = OM}$$

$$\& \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\therefore \tan\phi = \lim_{\delta\theta \to 0} \frac{(r+\delta r)\left(\delta\theta - \frac{\delta\theta^3}{3!} + \frac{\delta\theta^5}{5!} - \cdots\right)}{(r+\delta r)\left(1 - \frac{\delta\theta^2}{2!} + \frac{\delta\theta^4}{4!} - \cdots\right) - r}$$

[Neglecting higher power of $\delta\theta$, we get]

$$\tan\phi = \lim_{\delta\theta \to 0} \frac{(r+\delta r)\,\delta\theta}{(r+\delta r).1 - r}$$

$$\Rightarrow \tan\phi = \lim_{\delta\theta \to 0} \frac{r\delta\theta + \delta r.\delta\theta}{r + \delta\theta - r} = \lim_{\delta\theta \to 0}$$

Neglecting $\delta r\delta\theta$, (because $\delta r.\delta\theta$ is very-very small) we get.

$$\Rightarrow \tan\phi = \lim_{\delta\theta \to 0} \frac{r.\delta\theta}{\delta r}.$$

$$\Rightarrow \boxed{\tan\phi = \frac{r\,d\theta}{dr}}$$ (which is the required result.)

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(11)

Note!! Since, $\tan\phi = \frac{r\,d\theta}{dr}$, $\Rightarrow \boxed{\frac{\sin\phi}{\cos\phi} = \frac{r\,d\theta/ds}{dr/ds}}$

$\therefore \cot\phi = \frac{1}{r}\cdot\frac{dr}{d\theta}$

$\Rightarrow \boxed{\frac{\cos\phi}{\sin\phi} = \frac{1}{r}\cdot\frac{dr/ds}{d\theta/ds}}$.

$\therefore \boxed{\sin\phi = \frac{r\,d\theta}{ds} = r\dot{\theta}}$

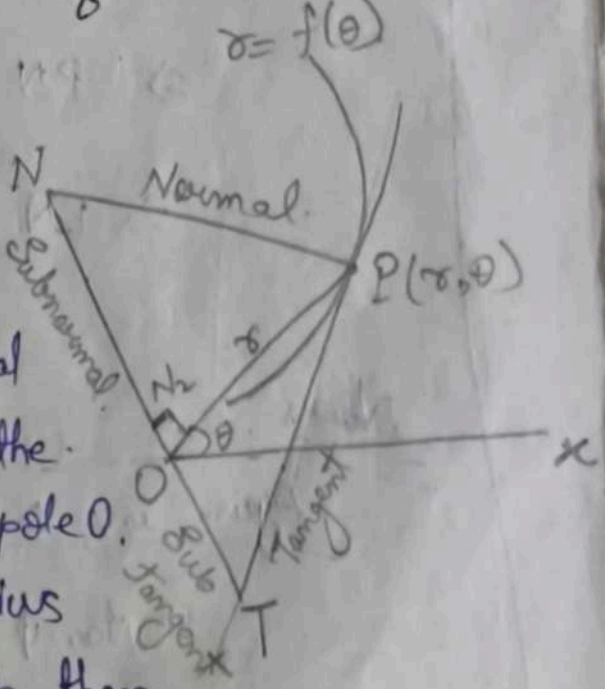
& $\boxed{\cos\phi = \frac{dr}{ds} = \dot{r}}$

Q. Find the polar length of tangent, normal, subtangent subnormal, in cartesian form ($r\theta$)

$\Rightarrow$ Soln:-

Let $P r\theta$ be any point on the curve $r = f(\theta)$.

Let the tangent & normal at P to the curve meets the straight line through the pole O at right angles to the radius vector OP. in T & N resp. then,



OT is called the length of polar sub tange. & ON is called polar sub normal. TN is called a length of polar normal & PT is called the length of polar tangent.

Now, $\boxed{OP = r}$, $\boxed{\angle OPT = \phi}$

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(2)

From the fig. it is evidence that,

$\angle PNO = \phi$ & $\angle POX = \theta$.

In right angled triangle PON,

$\cos\phi = \tan\phi = \frac{OP}{ON} = \frac{r}{ON}$ [i.e., the polar su

$\Rightarrow ON = \frac{r}{\tan\phi} = \frac{r}{r\frac{d\theta}{dr}} = \frac{dr}{d\theta}$

$\Rightarrow \boxed{ON = \frac{dr}{d\theta}}$ [$\because \tan\phi = \frac{r d\theta}{dr}$

[i.e., the polar sub normal = ON = $\frac{d}{d}$

Again,

$\sin\phi = \frac{OP}{PN} = \frac{r}{PN}$

$\Rightarrow PN = \frac{r}{\sin\phi} = r\operatorname{cosec}\phi$

$\Rightarrow PN = r\sqrt{1+\cot^2\theta} = r\sqrt{1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2}$

(which is the length of polar normal

Now,

Again in ΔOPT,

$\tan\phi = \frac{OT}{OP} = \frac{OT}{r}$

$\Rightarrow OT = r\tan\phi = r \cdot r \cdot \frac{d\theta}{dr} = r^2\frac{d\theta}{dr}$

$\Rightarrow \boxed{OT = r^2\frac{d\theta}{dr}}$ (which is called polar sub tangent)

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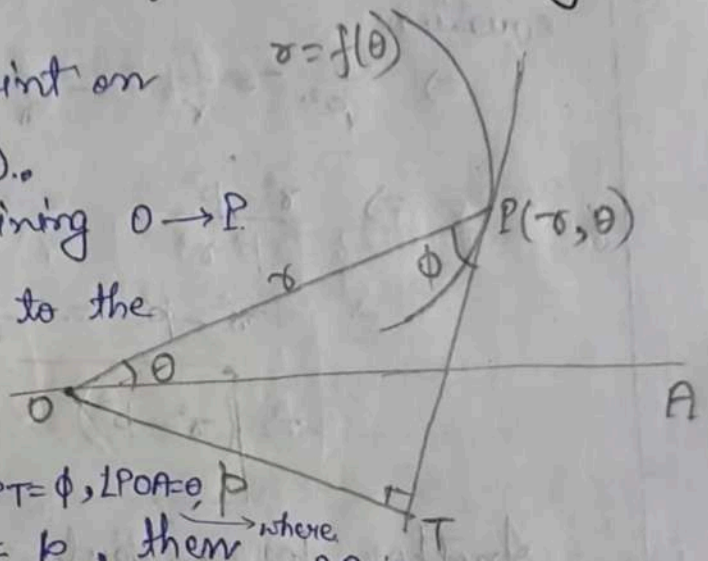
To find the perpendicular from pole on tangent

Let $P r\theta$ be any point on the curve $r = f(\theta)$..

'O' is the pole joining $O \to P$

From 'O', draw $\perp^r$ to the tangent at P to the curve.



$\angle OPT = \phi$, $\angle POA = \theta$, where OA is initial line.

Let $OP = r$, $OT = p$, then

right angled Δ OTP, $\sin\phi =$

$$\sin\phi = \frac{OT}{OP} = \frac{p}{r} \Rightarrow \boxed{p = r\sin\phi} \quad \text{— (i)}$$

which is the length of perpendicular from pole on tangent.

If we required the value of 'p' in terms of r & θ, then from equ (i)

$$p = r\sin\phi$$

$$\Rightarrow \frac{1}{p} = \frac{1}{r\sin\phi}$$

Squaring both sides, we get

$$\frac{1}{p^2} = \frac{1}{r^2\sin^2\phi} \Rightarrow \frac{1}{r^2}\operatorname{cosec}^2\phi.$$

Therefore, $\boxed{\frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2\phi)}$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right]$$

$$\Rightarrow \boxed{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2} \quad \text{— (ii)}$$

$\left[\because \tan\phi = \frac{r}{\frac{dr}{d\theta}} \Rightarrow \cot\phi = \frac{1}{r}\right]$

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which is called peddle **pedal equo** from polar equation

$$\Rightarrow \frac{1}{p^2} - \frac{1}{r^2} = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow \frac{r^2 - p^2}{p^2 r^2} = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow r^2 - p^2 = \frac{p^2}{r^2}\left(\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow \boxed{\left(r^2 - p^2\right) = \left(\frac{p}{r} \cdot \frac{dr}{d\theta}\right)^2}$$

Another form, put

$$\boxed{\frac{1}{r} = u} \Rightarrow \boxed{\frac{1}{r^2} = u^2}$$

$$\therefore \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

After squaring,

$$\boxed{\left(\frac{du}{d\theta}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}$$

Then from equo (ii),

$$\boxed{\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2} \quad \text{— (iii)}$$

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Q Find the pedal. (p-r) equ^{n} of the following curve.

Equ^{n} of curve, $\boxed{r^n = a^n \cos n\theta}$

⇒ Sol^{n}:- The given curve is $\boxed{r^n = a^n \cos n\theta}$ — (i)

Taking log both sides, we get

$$\boxed{n \log r = n \log a + \log \cos(n\theta)}$$

Diff. both sides, ~~we get~~ w.r.t. 'θ', we get

$$n \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = n \cdot \frac{1}{a} \cdot 0 + \frac{1}{\cos n\theta} (-\sin n\theta) \cdot n.$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = -\frac{\sin n\theta}{\cos n\theta}.$$

So, $\boxed{\frac{dr}{d\theta} = -r \tan(n\theta)}$ — (ii)

On squaring both sides, we get

$$\boxed{\left(\frac{dr}{d\theta}\right)^2 = r^2 \tan^2 n\theta} \quad \text{— (iii)}$$

Now, from Pedal equ^{n},

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \cdot r^2 \tan^2 n\theta$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left(1 + \tan^2 n\theta\right).$$

$$\Rightarrow \boxed{\frac{1}{p^2} = \frac{1}{r^2} \cdot \sec^2 n\theta} \quad \text{— (iv)}$$

Now,

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From equ (i), $\frac{r^n}{a^n} = \cos n\theta \Rightarrow \frac{a^n}{r^n} = \sec n\theta$

Squaring both sides,

$$\left(\frac{a^n}{r^n}\right)^2 = \sec^2 n\theta \quad \text{——— (v)}$$

Putting value of (v) in (iv), we get

$$\frac{1}{p^2} = \frac{1}{r^2}\left(\frac{a^{2n}}{r^{2n}}\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{a^{2n}}{r^{2n+2}}$$

$$\Rightarrow p^2 = \frac{r^{2(n+1)}}{a^{2n}} \Rightarrow p = \sqrt{\frac{r^{2(n+1)}}{a^{2n}}}$$

$$\Rightarrow p = \frac{r^{(n+1)}}{a^n} \Rightarrow r^{n+1} = p\,a^n$$

Ans

V.V.I.

(2.) Find the pedal equ from the following curve.

(i) $r = a\cos\theta$

(ii) $r^2 = a^2\cos 2\theta$

(iii) $r^3 = a^3\cos 3\theta$

(iv) $r^4 = a^4\cos 4\theta$

(v) $r^5 = a^5\cos 5\theta$

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3

Again, $\cos\phi = \frac{OP}{PT} \Rightarrow PT = \frac{r}{\cos\phi}$ [$\because OP = r$]

$\Rightarrow PT = r\sec\phi = r\sqrt{1+\tan^2\phi}$

$\Rightarrow PT = r\sqrt{1+\tan^2\phi} = r\sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}$

$\Rightarrow \boxed{PT = r\sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}}$ (which is the required length of polar tangent).

VI

Partial differentiation:-

Q. If $u = f(r)$, where $r^2 = x^2+y^2$, show that

$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2} = f''(r) + \frac{1}{r}f'(r)$.

Proof:- Since, $u = f(r)$, where $(r^2 = x^2+y^2)$ — (1)

Diff. equ (i) partially w.r.t. x,

$2r\cdot\frac{\delta r}{\delta x} = 2x+0$

$\Rightarrow \boxed{\frac{\delta r}{\delta x} = \frac{x}{r}}$ — (1a)

Ily $\boxed{\frac{\delta r}{\delta y} = \frac{y}{r}}$ — (1b)

Since, $\boxed{u = f(r)}$.

$\therefore$ Diff. partially w.r.t. x, we get

$\frac{\delta u}{\delta x} = f'(r)\cdot\frac{\delta r}{\delta x} = f'(r)\cdot\frac{x}{r}$ [using (1a)]

$\Rightarrow \boxed{\frac{\delta u}{\delta x} = f'(r)\frac{x}{r}}$

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Again diff. both sides w.r.t. x, we get

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{x f'(r)}{r}\right]$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{r \frac{\partial}{\partial x}\cdot[x \cdot f'(r)] - [x f'(r) \times \frac{\partial r}{\partial x}]}{r^2}$$

$$= \frac{r \times [1 \cdot f'(r) + x f''(r) \cdot \frac{\partial r}{\partial x}] - x f'(r) \cdot \frac{x}{r}}{r^2}$$

$$= \frac{1}{r^2}\left[r f'(r) + r \cdot x f''(r) \cdot \frac{x}{r} - \frac{x^2}{r} f'(r)\right]$$

[using $\frac{\partial r}{\partial x} = \frac{x}{r}$]

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{r^2}\left[x^2 f''(r) + r f'(r) - \frac{x^2}{r} f'(r)\right]} \quad (2)$$

Again, since, $u = f(r)$.

Diff. partially w.r.t. 'y'

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$$\boxed{\frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[y^2 f''(r) + r f'(r) - \frac{y^2}{r} f'(r)\right]} \quad \text{--- (3)}$$

From (ii) Adding (2) & (3),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[x^2 f''(r) + r f'(r) - \frac{x^2}{r} f'(r) + y^2 f''(r) - r f'(r) - \frac{y^2}{r} f'(r)\right]$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[(x^2+y^2) f''(r) + 2r f'(r) - \left(\frac{x^2+y^2}{r}\right) f'(r)\right]$$

From using equ (i),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[r^2 f''(r) + 2r f'(r) - \frac{r^2}{r} f'(r)\right]$$

$$= f''(r) + \frac{r f'(r)}{r^2}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)}$$ proved

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Partial Differentiation

Let 'u' be a symbol of which has definite value for every pair of values of x & y, then 'u' is called a function of independent variables 'x' & 'y' is written as -

$$u = f(x,y) = u(x,y)$$

⇒ Partial Differential Calculus Coefficients:

The Partial Differential Coefficient of $f(x,y)$ w.r.t 'x' is defined as :-

$$\lim_{\delta x \to 0} \frac{\delta u}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

provided these limits exist and is written as -

$$\frac{\delta f}{\delta x} \text{ or } f(x) \text{ or } Dxf$$

Similarly, the partial differential coefficient of $f(x,y)$ w.r.t. y is defined as -

$$\lim_{\delta y \to 0} \frac{\delta u}{\delta y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

provided these limits exist & is written as -

$$\frac{\delta f}{\delta y} \text{ or } f(y) \text{ or } Dyf$$

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So, Partial Differential coefficient (P.D.C.) of $f(x,y)$ w.r.t. x is the ordinary differential coefficient of $f(x,y)$ where y is regarded as constant.

Similarly, the P.D.C. of $f(x,y)$ w.r.t. y is the ordinary differential coefficient of $f(x,y)$ when x is regarded as constant.

We can further differentiate $f(x)$ & $f(y)$ w.r.t. 'x' & 'y' partially.

The P.D.C. of $\frac{\delta f}{\delta x}$ partially w.r.t. x & y resp.

$$\frac{\delta}{\delta x}\left(\frac{\delta f}{\delta x}\right) = \frac{\delta^2 f}{\delta x^2}$$

$$\frac{\delta}{\delta y}\left(\frac{\delta f}{\delta x}\right) = \frac{\delta^2 f}{\delta y \delta x}$$

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$u = x^3 + 3x^2y + 3xy^2 + y^3$.

Verify that $\frac{\delta^2 f}{\delta x \delta y} = \frac{\delta^2 f}{\delta y \delta x}$, when $u = x\sin y + y\sin x$.

Soln:-

Given:- $u = x\sin y + y\sin x$ — (i)

Diff. equ (i) partially w.r.t. x.

$$\frac{\delta u}{\delta x} = (1.\sin y + x.0) + (0.\sin x + y\cos x).$$

$$\Rightarrow \frac{\delta u}{\delta x} = \sin y + y\cos x \quad \text{— (ii)}$$

Again diff. equ (ii) partially w.r.t. y, we get

$$\frac{\delta}{\delta y}\left(\frac{\delta u}{\delta x}\right) = \frac{\delta}{\delta y}(\sin y + y\cos x).$$

$$\frac{\delta^2 u}{\delta y \delta x} = \cos y + 1.\cos x + 0$$

$$\frac{\delta^2 u}{\delta y \delta x} = 1.\cos x + \cos y \quad \text{— (iii)}$$

Now, diff. equ (i) partially w.r.t. y.

$$\frac{\delta u}{\delta y} = x\cos y + \sin x \quad \text{— (iv)}$$

Again diff. equ (iv) w.r.t. x partially,

$$\frac{\delta}{\delta x}\left(\frac{\delta u}{\delta y}\right) = \frac{\delta}{\delta x}[x\cos y + \sin x].$$

$$\frac{\delta^2 u}{\delta x \delta y} = 1.\cos y + 0 + \cos x.$$

$$\Rightarrow \frac{\delta^2 u}{\delta x \delta y} = \cos x + \cos y. \quad \text{— (v)}$$

Hence, from equ (iii) & (v), $\boxed{\frac{\delta^2 f}{\delta y \delta x} = \frac{\delta^2 f}{\delta x \delta y}}$

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8). Verify $\frac{\delta^2 f}{\delta x \delta y} = \frac{\delta^2 f}{\delta y \delta x}$ when

(i) $u = x^y$

(ii) $u = \log\left(\frac{x^2+y^2}{xy}\right)$

⇒ Soln:-

(ii) $u = x^y$

Taking log on both sides.

$\log u = y \log x$

$\Rightarrow u = e^{y \log x}$ — (i)

$\therefore u = x^y$

Diff. $\log u = y \log x$ w.r.t. x. we get

$\frac{1}{u} \frac{\delta u}{\delta x} = \frac{y}{x}$ $(\because x^y)$

$\frac{\delta u}{\delta x} = \frac{x^y \cdot y}{x}$

Now, w.r.t. 'y'

$\frac{\delta^2 u}{\delta y \delta x} = \frac{u \cdot 1}{x} = \frac{u}{x}$ — (iii)

$\log u = y \log x$.

w.r.t. 'y'

$\frac{1}{u} \cdot \frac{\delta u}{\delta} = \log x = \frac{\delta u}{\delta y} = u \log x$.

w.r.t. 'x',

$\frac{\delta^2 u}{\delta x \delta y} = \frac{u}{x}$ — (ii)

Hence, from equ (i) & (ii),

$$\boxed{\frac{\delta^2 f}{\delta y \delta x} = \frac{\delta^2 f}{\delta x \delta y}}$$ proved

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(ii) $u = \log(x^2+y^2) - \log x - \log y$ — (i)

Diff. equo (i) partially w.r.t. x,

$$\frac{\delta u}{\delta x} = \frac{2x}{x^2+y^2} - \frac{1}{x} \quad \text{— (ii)}$$

$$\frac{\delta^2 u}{\delta y \delta x} = -2x2y(x^2+y^2)^{-2}$$

$$\Rightarrow \frac{\delta^2 u}{\delta y \delta x} = \frac{-4xy}{(x^2+y^2)^2} \quad \text{— (iii)}$$

Diff. equo (i) partially w.r.t. y.

$$\frac{\delta u}{\delta y} = \frac{2y}{x^2+y^2} - \frac{1}{y} \quad \text{— (iv)}$$

$$\frac{\delta^2 u}{\delta x \delta y} = -2x2y(x^2+y^2)^{-2}$$

$$\Rightarrow \frac{\delta^2 u}{\delta x \delta y} = \frac{-4xy}{(x^2+y^2)^2} \quad \text{— (v)}$$

Hence, from equo (iii) & (v)

$$\boxed{\frac{\delta^2 u}{\delta x \delta y} = \frac{\delta^2 u}{\delta y \delta x}}$$ proved

4) Verify $\frac{\delta^2 f}{\delta x \delta y} = \frac{\delta^2 f}{\delta y \delta x}$ if $u = \tan^{-1}\left(\frac{y}{x}\right)$

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(5) If $u = x^2y + y^2z + z^2x$, then show that

$$\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} + \frac{\delta u}{\delta z} = (x+y+z)^2.$$

⇒ Soln:— $u = x^2y + y^2z + z^2x$ —— (given)

Diff. both sides w.r.t. x partially,

$$\frac{\delta u}{\delta x} = 2xy + 0 + z^2 = 2xy + z^2 \quad \text{—— (i)}$$

Similarly, $\frac{\delta u}{\delta y} = x^2 + 2yz$ —— (ii)

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& $\frac{\delta u}{\delta z} = y^2 + 2xz$ —— (iii)

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From eqn (i), (ii) & (iii)

$$\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} + \frac{\delta u}{\delta z} = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$

Hence, $\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} + \frac{\delta u}{\delta z} = (x+y+z)^2$ proved

(6.) If $r = (x^2+y^2+z^2)^{-1/2}$, show that $\frac{\delta^2 r}{\delta x^2} + \frac{\delta^2 r}{\delta y^2} + \frac{\delta^2 r}{\delta z^2} = 0$

(7.) If $u = e^{xyz}$, show that

$$\frac{\delta^3 u}{\delta x \delta y \delta z} = (1 + 3xyz + x^2y^2z^2)\, e^{xyz}.$$

(8.) If $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, prove that

$$\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} + \frac{\delta u}{\delta z} = 0.$$

(9.) If $u = f(r)$, where $r^2 = x^2 + y^2$, show that.

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = f''(r) + \frac{1}{r} f'(r).$$

(10.) If $r = (x^2+y^2+z^2)^{-1/2}$, show that

$$\frac{\delta^2 r}{\delta x^2} + \frac{\delta^2 r}{\delta y^2} + \frac{\delta^2 r}{\delta z^2} = \frac{2}{r}.$$

(11.) If $u = \log(x^3+y^3+z^3-3xyz)$, show that.

$$\left(\frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

Sol^n:- Since, we have given, $u = \log(x^3+y^3+z^3-3xyz)$ → (i)

Diff. eqn (i) partially w.r.t. x, we get.

$$\frac{\delta u}{\delta x} = \frac{1}{(x^3+y^3+z^3-3xyz)}(3x^2 - 3yz) \quad \text{— (ii)}$$

Ily, $\frac{\delta u}{\delta y} = \frac{1}{(x^3+y^3+z^3-3xyz)} \times (3y^2 - 3xz)$ — (iii)

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Q. If $x^x \cdot y^y \cdot z^z = c$, show that at $x = y = z$,

$$\frac{\delta^2 z}{\delta x \delta y} = -[x \log_e x]^{-1}$$

⇒

Proof :- We have, $x^x \cdot y^y \cdot z^z = c$.

Taking log both sides, we get

$$x \log x + y \log y + z \log z = \log c \quad \longrightarrow (i)$$

$$\Rightarrow x \log x + y \log y + z \log z = c$$

Diff. both sides partially w.r.t. x, we get

$$1 \cdot \log x + x \cdot \frac{1}{x} + 0 + \frac{\delta z}{\delta x} \cdot \log z + z \cdot \frac{1}{z} \cdot \frac{\delta z}{\delta x} = 0$$

$$\Rightarrow (1 + \log x) + (1 + \log z) \frac{\delta z}{\delta x} = 0.$$

$$\Rightarrow (1 + \log x) = -(1 + \log z) \frac{\delta z}{\delta x}$$

$$\Rightarrow \boxed{\frac{\delta z}{\delta x} = -\left(\frac{1 + \log x}{1 + \log z}\right)} \quad \text{—— (ii)}$$

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Again, diff equ (i) partially w.r.t. y, we get

$$0 + 1 \cdot \log y + y \cdot \frac{1}{y} + \frac{\delta z}{\delta y} \cdot \log z + z \cdot \frac{1}{z} \cdot \frac{\delta z}{\delta y} = 0$$

$$\Rightarrow (1 + \log y) = -(1 + \log z) \cdot \frac{\delta z}{\delta y}$$

$$\Rightarrow \boxed{\frac{\delta z}{\delta y} = -\left(\frac{1 + \log y}{1 + \log z}\right)} \quad \text{—— (iii)}$$

Diff. equ (iii) partially w.r.t. x, we get

$$\frac{\delta}{\delta x}\left(\frac{\delta z}{\delta y}\right) = \frac{-\delta}{\delta x}\left(\frac{1 + \log y}{1 + \log z}\right) \quad \text{—— (iv)}$$

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$$\frac{\delta^2 z}{\delta x \delta y} = -\left[\frac{0-(1+\log y)\times\left(0+\frac{1}{z}\cdot\frac{\delta z}{\delta x}\right)}{(1+\log z)^2}\right]$$

$$\Rightarrow \frac{\delta^2 z}{\delta x \delta y} = \frac{\frac{1}{z}(1+\log y)\frac{\delta z}{\delta x}}{(1+\log z)^2} = \frac{1}{z}\frac{(1+\log y)}{(1+\log z)^2} \times \frac{(-)(1+\log y)}{(1+\log z)}$$

$$\Rightarrow \frac{\delta^2 z}{\delta x \delta y} = -\frac{1}{x}\frac{(1+\log x)}{(1+\log x)^2} \times \frac{(1+\log x)}{(1+\log x)}$$

$$\Rightarrow \frac{\delta^2 z}{\delta x \delta y} = -\frac{1}{x}\frac{1}{(1+\log x)} \quad [\because \log e = 1]$$

$$\Rightarrow \frac{\delta^2 z}{\delta x \delta y} = -\frac{1}{x}\frac{1}{(\log e + \log x)}$$

$$\Rightarrow \boxed{\frac{\delta^2 z}{\delta x \delta y} = -\frac{1}{x \log ex} = -[x \log ex]^{-1}} \quad \text{proved}$$

VI. 2.8 If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, show that

$$\left(\frac{\delta u}{\delta x}\right)^2 + \left(\frac{\delta u}{\delta y}\right)^2 + \left(\frac{\delta u}{\delta z}\right)^2 = 2\left(x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} + z\frac{\delta u}{\delta z}\right)$$

or.

$$\boxed{u_x^2 + u_y^2 + u_z^2 = 2[x u_x + y u_y + z u_z]}$$

Proof:- We have, $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$ ——— (i)

Diff. equ (i) partially w.r.t. x, we get

$$\frac{(2x)(a^2+u) - x^2\left(0+\frac{\delta u}{\delta x}\right)}{(a^2+u)^2} + \frac{0 - y^2\left(0+\frac{\delta u}{\delta x}\right)}{(b^2+u)^2} + \frac{0 - z^2\left(0+\frac{\delta u}{\delta x}\right)}{(c^2+u)^2} = 0$$

$$\Rightarrow \frac{2x(a^2+u)}{(a^2+u)^2} - \frac{x^2\frac{\delta u}{\delta x}}{(a^2+u)^2} - \frac{y^2\frac{\delta u}{\delta x}}{(b^2+u)^2} - \frac{z^2\frac{\delta u}{\delta x}}{(c^2+u)^2} = 0$$

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$$\Rightarrow \frac{2x(a^2+u)}{(a^2+u)^2} = \frac{\delta u}{\delta x}\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$$

$$\Rightarrow \frac{2x}{(a^2+u)} = \frac{\delta u}{\delta x}\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$$

$$\Rightarrow \frac{\delta u}{\delta x} = \left(\frac{2x}{a^2+u}\right) \Big/ \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] \text{---}$$

Similarly, diff. equo (i) partially w.r.t. y & z we get

$$\frac{\delta u}{\delta y} = \left[\frac{2y}{(b^2+u)}\right] \Big/ \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] \text{---}$$

& $$\frac{\delta u}{\delta z} = \left[\frac{2z}{(c^2+u)}\right] \Big/ \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] \text{---}$$

Now, Squaring equo (ii), (iii) & (iv), we get

$$\left(\frac{\delta u}{\delta x}\right)^2 = \left(\frac{4x^2}{(a^2+u)^2}\right) \Big/ \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]^2 \text{---} (iii)$$

$$\left(\frac{\delta u}{\delta y}\right)^2 = \left[\frac{4y^2}{(b^2+u)^2}\right] \Big/ \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]^2 \text{---} (iii)$$

& $$\left(\frac{\delta u}{\delta z}\right)^2 = \left[\frac{4z^2}{(c^2+u)^2}\right] \Big/ \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]^2 \text{---} (iv)$$

Now, adding [(ii)-a], [(iii)-a] & [(iv)-a], we get

$$\left(\frac{\delta u}{\delta x}\right)^2+\left(\frac{\delta u}{\delta y}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2 = 4\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]$$

$$\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]^2$$

$$\Rightarrow \left(\frac{\delta u}{\delta x}\right)^2+\left(\frac{\delta u}{\delta y}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2 = \frac{4}{\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]}$$

$$\Rightarrow \left(\frac{\delta u}{\delta x}\right)^2+\left(\frac{\delta u}{\delta y}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2 = 2\left\{\frac{2}{\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]}\right\} \quad \text{(v)}$$

Now, equo (iii), (iii) & (iv) multiplying by x, y & z resp. we get.

$$x\left(\frac{\delta u}{\delta x}\right) = \left[\frac{2x^2}{(a^2+u)}\right] \Big/ \left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right] \quad \text{(iii-b)}$$

$$y\left(\frac{\delta u}{\delta y}\right) = \left[\frac{2y^2}{(b^2+u)^2}\right] \Big/ \left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right] \quad \text{(iii-b)}$$

$$\& \quad z\left(\frac{\delta u}{\delta z}\right) = \left[\frac{2z^2}{(c^2+u)^2}\right] \Big/ \left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right] \quad \text{(iv-b)}$$

Adding equo [(iii)-b], [(iii)-b] & [(iv)-b], we get

$$x\left(\frac{\delta u}{\delta x}\right)+y\left(\frac{\delta u}{\delta y}\right)+z\left(\frac{\delta u}{\delta z}\right) = \left[\frac{2x^2}{(a^2+u)}+\frac{2y^2}{(b^2+u)}+\frac{2z^2}{(c^2+u)}\right] \Big/ \left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]$$

$$\Rightarrow x\left(\frac{\delta u}{\delta x}\right)+y\left(\frac{\delta u}{\delta y}\right)+z\left(\frac{\delta u}{\delta z}\right) = 2\left[\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right] \Big/ \left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]$$

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Using equo (1), we get.

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} + z\frac{\delta u}{\delta z} = \frac{2}{\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right.}$$

Putting value of (vi), in (v), we get,

Putting value of

$$\left(\frac{\delta u}{\delta x}\right)^2 + \left(\frac{\delta u}{\delta y}\right)^2 + \left(\frac{\delta u}{\delta z}\right)^2 = 2\left[x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} + z\right.$$

proved.

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Total Differentiation

(i) If $u = f(x,y)$, where $x = \phi(t)$, $y = \psi(t)$, then 'u' can be expressed as a fn of 't' along by substituting the values of x & y in terms of 't' from the last 2 equ^{n}, in the first two equations. And also, we can find the ordinary differential coefficient (du/dt), which is called the <u>total differential coefficient of 'u' w.r.t. 't'</u>.

Here, 'u' is also called <u>composite function of 't'</u>.

(ii) $z = f(x,y)$, $x = \phi(u,v)$, $y = \psi(u,v)$.

diff. of composite fn :-

$$\delta z = \frac{\partial f}{\partial x}\cdot \delta x + \frac{\partial f}{\partial y}\cdot \delta y.$$

$$\Rightarrow \frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\delta z = \frac{\partial z}{\partial x}\cdot \delta x + \frac{\partial z}{\partial y}\cdot \delta y.$$

$$\Rightarrow \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

(i) $u = f(x,y)$.

$$\delta u = \frac{\partial f}{\partial x}\cdot \delta x + \frac{\partial f}{\partial y}\cdot \delta y.$$

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}.$$

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1. Q. If z is a fn of (x, y), where, $x = e^u + e^v$, $y = e^u - e^v$, show that $\boxed{\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}}$.

⇒ Proof:- ∵ We have

z is a composite fn of u & v, then diff of composite fn be.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \quad \text{— (i)}$$

& $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} \quad \text{— (ii)}$$

Now, we have, $\boxed{x = e^u + e^v}$ & $\boxed{y = e^u - e^v}$

$\therefore \boxed{\frac{\partial x}{\partial u} = e^u, \quad \frac{\partial x}{\partial v} = -e^v}$, Ily, $\boxed{\frac{\partial y}{\partial u} = -e^u, \quad \frac{\partial y}{\partial v} = -e^v}$

From equ (i), $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot e^u + \frac{\partial z}{\partial y} \cdot (-e^u)$ — (iii)

From equ (ii), $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot (-e^v) + \frac{\partial z}{\partial y} \cdot (-e^v)$ — (

equ$_s$ (iii) − (iv).

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} e^u - \frac{\partial z}{\partial y} e^{-u} + \frac{\partial z}{\partial x} e^{-v} + \frac{\partial z}{\partial y} e^v$$

$$\Rightarrow \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^u + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} + e^v)\frac{\partial z}{\partial y}$$

⇒ So, from equ,

$$\boxed{\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}}$$

proved

H.W

(2) Find du/dt, when $u = x^2 + y^2$, $x = at^2$, $y = 2at$. verify by direct substitution.

(3) If $z = \sqrt{x^2+y^2}$ & $x^3 + y^3 + 3axy = 5a^2$, find the value of $\frac{dz}{dx}$ when $x = a$ & $y = a$.

W1.

(4) If $V = f(y-z, z-x, x-y)$, prove that

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z} = 0.$$

Diff.

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Differentiation of Implicit function

→ The relations $f(x,y) = c$, where 'y' denotes explicit in terms of x are called implicit functions and such equations defined 'y' as an implicit fn of x.

I.B. To find dy/dx & d^2y/dx^2 for any implicit equn $f(x,y) = 0$ or $f(x,y) = c$.

⇒ **Soln:-** ∵ $f(x,y)$ is a fn of two variables x & y and 'y' itself is a function of x, so that we may consider $f(x,y)$ as a composite fn. of x. So, its derivatives w.r.t. x is

i.e., $df = \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$.

i.e., $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{\partial f}{\partial x} = -\frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$.

$\Rightarrow \frac{\partial f}{\partial x} = \frac{-\partial f/\partial x}{\partial f/\partial y}$

$\therefore \frac{\partial y}{\partial x} = -\frac{f_x}{f_y}$, where $f_y \neq 0$.

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Taylor's Expansion

State & prove Taylor's theorem in 2 variables.

⇒ **Statement :-** $f(x,y)$ & all its partial derivatives up to order 'n' be continuous in all neighbourhoods of the point (x,y), then,

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial y\,\partial x} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \frac{1}{3!}\left[h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2\partial y} + 3hk^2\frac{\partial^3 f}{\partial x\,\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}\right) + \frac{1}{4!}[\ldots] + \ldots$$

or

$$\Rightarrow f(x+h, y+k) = f(x,y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f + \frac{1}{3!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^3 f + \frac{1}{4!}[\ldots]^4 f + \ldots$$

⇒ **Proof :-** We know that by Taylor's theorem by a function of single variable x,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots$$

$$\Rightarrow f(x+h) = f(x) + h\frac{d}{dx}[f(x)] + \frac{h^2}{2!}\frac{d^2}{dx^2}[f(x)] + \frac{h^3}{3!}\frac{d^3}{dx^3}[f(x)] + \ldots$$

or by partially

$$\Rightarrow f(x+h) = f(x) + h\frac{\partial}{\partial x}f(x) + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}f(x) + \frac{h^3}{3!}\frac{\partial^3}{\partial x^3}[f(x)] + \ldots \quad \rightarrow (1)$$

Now, treating 'y' and hence $(y+k)$ as a constant, then by Taylor's theorem for a fn of single variable 'x' from equ (i), we get

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$$\Rightarrow \boxed{f(x+h, y+k) = f(x, y+k) + h\frac{\partial}{\partial x} f(x, y+k) + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2} f(x, y+k) + \frac{h^3}{3!}\frac{\partial^3}{\partial x^3} f(x, y+k) + \cdots \cdots} \quad \text{— (2)}$$

Again, keeping 'x' as a constant and applying Taylor's theorem for a fn of single variable 'y', we get

$$\boxed{f(y+k) = f(y) + k\frac{\partial}{\partial y} f(y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(y) + \frac{k^3}{3!}\frac{\partial^3}{\partial y^3} f(y) + \cdots} \quad \text{— (3)}$$

Putting the value of $f(x, y+k)$ from equo (3) in equo (2), we get

$$\begin{aligned} f(x+h, y+k) = {} & \left[f(x,y) + k\frac{\partial}{\partial y} f(x,y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x,y) + \cdots\right] \\ & + h\frac{\partial}{\partial x}\left[f(x,y) + k\frac{\partial}{\partial y} f(x,y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x,y) + \cdots\right] \\ & + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left[f(x,y) + k\frac{\partial}{\partial y} f(x,y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x,y) + \cdots\right] \\ & + \frac{h^3}{3!}\frac{\partial^2}{\partial x^2}\left[f(x,y) + k\frac{\partial}{\partial y} f(x,y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x,y) + \cdots\right] \\ & + \cdots \cdots \cdots \end{aligned}$$

Replacing $\boxed{[f(x,y) \longrightarrow f]}$, in all partial derivatives,

$$\begin{aligned} \Rightarrow f(x+h, y+k) = {} & \left[f + k\frac{\partial}{\partial y} f + \frac{k^2}{2!}\frac{\partial^2}{\partial y} f + \frac{k^3}{3!}\frac{\partial^3}{\partial y^3} f + \cdots\right] \\ & + \frac{h\partial}{\partial x}\left[f + k\frac{\partial}{\partial y} f + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f + \frac{k^3}{3!}\frac{\partial^3}{\partial y^3} f + \cdots\right] \\ & + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left[f + k\frac{\partial}{\partial y} f + \frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2} + \frac{k^3}{3!}\frac{\partial^3}{\partial y^3} f + \cdots\right] \\ & + \frac{h^3}{3!}\frac{\partial^3}{\partial x^3}\left[f + k\frac{\partial}{\partial y} f + \frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2} + \frac{k^3}{3!}\frac{\partial^3}{\partial y^3} f + \cdots\right] \\ & + \cdots \cdots \cdots \end{aligned}$$

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opening the brackets,

$$f(x+h,y+k) = f + k\frac{\partial}{\partial y}f + \frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2}f + \frac{k^3}{3!}\frac{\partial^3}{\partial y^3}f + \cdots$$
$$+ h\frac{\partial}{\partial x}f + hk\frac{\partial^2 f}{\partial x\partial y} + h\frac{k^2}{2!}\frac{\partial^3 f}{\partial x\partial y^2} + \frac{hk^3}{3!}\frac{\partial^4 f}{\partial x\partial y^3} + \cdots$$
$$+ \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}f + \frac{h^2k}{2!}\frac{\partial^3 f}{\partial x^2\partial y} + \frac{h^2k^2}{2!2!}\frac{\partial^4 f}{\partial x^2\partial y^2} + \cdots]$$
$$+ \left\{\frac{h^3}{3!}\frac{\partial^3 f}{\partial x^3} + \frac{h^3k}{3!}\frac{\partial^4 f}{\partial x^3\partial y} + \cdots\right\} + \cdots$$

Now, grouping the terms of partial derivatives of same order in one brackets, we obtain,

$$f(x+h,y+k) = f(x,y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \left[\frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2} + hk\frac{\partial^2 f}{\partial x\partial y} + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f\right] + \left[\frac{h^3}{3!}\frac{\partial^3 f}{\partial x^3} + \frac{h^2k}{2!}\frac{\partial^3 f}{\partial x^2\partial y} + \frac{hk^2}{2!}\frac{\partial^3 f}{\partial x\partial y^2} + \frac{k^3}{3!}\frac{\partial^3 f}{\partial y^3}\right] + [\cdots] + \cdots$$

or,

$$f(x+h,y+k) = f(x,y) + \left[h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right] + \left[\frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2} + \frac{2}{2!}hk\frac{\partial^2 f}{\partial x\partial y} + \frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2}\right] + \left[\frac{h^3}{3!}\frac{\partial^3 f}{\partial x^3} + \frac{3h^2k}{3.2!}\frac{\partial^3 f}{\partial x^2\partial y} + \frac{3hk^2}{3.2!}\frac{\partial^3 f}{\partial x\partial y^2} + \frac{k^3}{3!}\frac{\partial^3 f}{\partial y^3}\right] + \cdots$$

$$\Rightarrow f(x+h,y+k) = f(x,y) + \left[h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right] + \left[\frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2} + \frac{2}{2!}hk\frac{\partial^2 f}{\partial x\partial y} + \frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2}\right] + \left[\frac{h^3}{3!}\frac{\partial^3 f}{\partial x^3} + \frac{3h^2k}{3!}\frac{\partial^3 f}{\partial x^2\partial y} + \frac{3hk^2}{3!}\frac{\partial^3 f}{\partial x\partial y^2} + \frac{k^3}{3!}\frac{\partial^3 f}{\partial y^3}\right] + \cdots$$

$$\Rightarrow f(x+h,y+k) = f(x,y) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]f + \left[\frac{h^2}{2!}\frac{\partial^2}{\partial x^2} + \frac{2}{2!}hk\frac{\partial^2}{\partial x\partial y} + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2}\right]f + \left[\frac{h^3}{3!}\frac{\partial^3}{\partial x^3} + \frac{3h^2k}{3!}\frac{\partial^3}{\partial x^2\partial y} + \frac{3hk^2}{3!}\frac{\partial^3}{\partial x\partial y^2} + \frac{k^3}{3!}\frac{\partial^3}{\partial y^3}\right]f + \cdots \quad (4)$$

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$$\Rightarrow f(x+h, y+k) = f(x,y) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]f + \frac{1}{2!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f + \frac{1}{3!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^3 f + \frac{1}{4!}[\text{---}] + \text{---} \quad \text{proved} \quad \text{---} (5)$$

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NOTE !! (ii) Expansion of $f(x,y)$ in powers of $(x-a)$ & $(x-b)$ in the neighbourhood of the point (a,b)

Putting $\boxed{a+h=x}$ & $\boxed{b+k=y}$

$\Rightarrow \boxed{h=x-a}$ $\Rightarrow \boxed{k=y-b}$

Now, from equo (5) & (7)

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \frac{1}{3!}[\text{---}] + \text{---} \quad (8)$$

Note (i) :- Expansion of $f(x,y)$ in neighbourhood of the point a,b — we know that $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x \partial y}$, ---- can be alternatively written as $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial^2 f}{\partial x^2} = f_{xx}$, $\frac{\partial f}{\partial y} = f_y$, $\frac{\partial^2 f}{\partial y^2} = f_{yy}$, --- resp. then from equo (4),

$$\boxed{f(x+h, y+k) = f(x,y) + (hf_x + kf_y) + \frac{1}{2!}\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right) + \frac{1}{3!}\left(h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}\right) + \text{---} \quad (6)}$$

Putting ($x=a$ & $y=b$) in equo (6), we get

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$$+\frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}[\text{---}] + \text{---} \quad \text{---(7)}$$

NOTE (iii) :- Expansion of $f(x,y)$ in powers of x & y in the neighbourhood of the point $(0,0)$.

Putting $(a=0$ & $b=0)$, in above equ (8), we get

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \frac{1}{3!}[\text{---}] + \text{---} \quad \text{---(9)}$$

Expand $\sin(x+h)(y+k)$ by Taylor's theorem as for as terms of 2nd degree.

Soln :- Let $f[(x+h), (y+k)] = \sin[(x+h)(y+k)]$.

$\therefore f(x,y) = \sin xy$.

So, $\frac{\partial f}{\partial x} = f_x = y\cos(xy)$

$\frac{\partial f}{\partial y} = f_y = x\cos(xy)$

$\frac{\partial^2 f}{\partial y^2} = f_{yy} = -x^2 \sin(xy)$

$\frac{\partial^2 f}{\partial x^2} = f_{xx} = -y^2 \sin(xy)$

$\frac{\partial^2 f}{\partial x \partial y} = f_{xy} = 1\cdot\cos xy - (-xy\sin xy)$

Then, by Taylor's theorem,

$$f(x+h, y+k) = f(x,y) + \left[h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right] + \frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right] + \text{---}$$

$$f(x+h, y+k) = \sin(x+h)(y+k) = \sin xy + [hy\cos xy + kx\cos xy] + \frac{1}{2!}[h^2 y^2 \sin xy + 2hk\{\cos xy - xy\sin xy\} - k^2 x^2 \sin xy] + \text{---}$$

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$\Rightarrow \boxed{\sin(x+h)(y+k) = \sin xy + [hy\cos xy + kx\cos xy] + \frac{1}{2!}[2hk\cos xy - \{h^2y^2 + 2hkxy + k^2y^2\}\sin xy] + \text{-----}}$ Ans

2. Q. Expand $\frac{(x+h)(y+k)}{x+h+y+k}$ in powers of $(h \& k)$ up to and inclusive of the 2nd degree terms.

$\Rightarrow$ Soln:-

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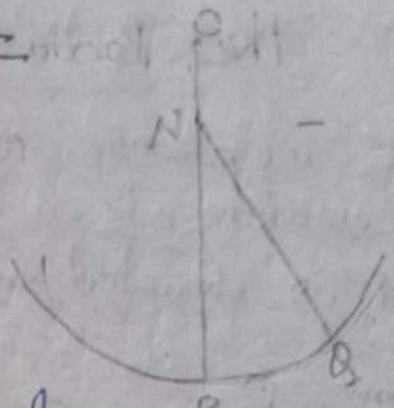
Curvature



Let P & Q be two neighbouring points on the curve.

& N be the point of intersection of normal to the curve at P & Q.

If N tends to ~~a diff~~ definite position C as Q tends to P. Then, C is defined as the centre of curvature of the curve at P.

The lengths CP is called the radius of curvature at P & it is generally denoted by 'ρ'.

The circle with centre 'C' & radius is called the circle of curvature at P.

The reciprocal of CP i.e., $\frac{1}{\rho}$ is called curvature at P.

A chord drawn through P of the circle of the curvature at P called the ~~curvature~~ curvature at P

In other words, curvature of a curve at a point in the arc rate of rotation of the tangent at that point, is denoted by the symbol K (kappa).

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The Radius of Curvature

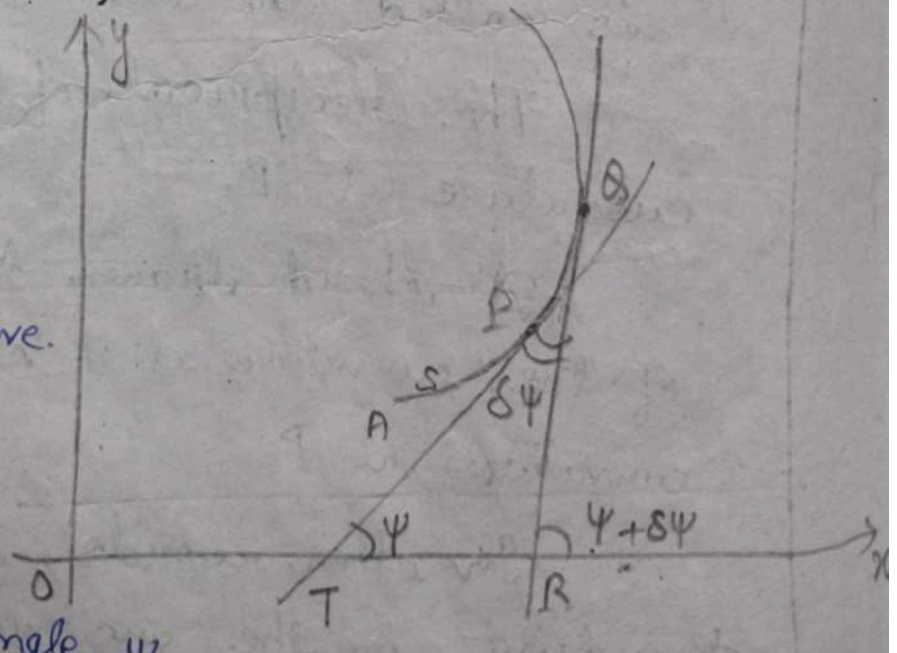
The reciprocal of curvature at a point is called the radius of curvature at that point.

It is denoted by the symbol 'ρ', i.e., $\boxed{\rho = \frac{1}{K}}$.

Form $\left\{\begin{array}{l} (x,y) \longrightarrow \text{cartesian form.} \\ (r,\theta) = \text{polar form} \\ (p,r) = \text{pedal form} \\ (p,\psi) = \text{polar tangential} \\ (s,\psi) = \text{Intrinsic form.} \end{array}\right\}$

Theorem ① :-

To find the formulae of radius of curvature in Intrinsic form (s,ψ).



Proof :-

Let P be any point on the curve where the arc $\boxed{\text{arc } AP = s}$

Let the tangential at P makes an angle ψ with x-axis.

Let Q be any other point on the curve

arc PQ = δs

Let the tangent at Q makes an angle $(\psi+\delta\psi)$ with x-axis. Then arc rate of rotation of the tangent at that point be

$$K = \lim_{\delta s \to 0} \frac{\delta\psi}{\delta s}$$

$$\therefore \boxed{K = \frac{d\psi}{ds}}$$

$$\because \rho = \frac{1}{K} = \frac{1}{\frac{d\psi}{ds}} \Rightarrow \boxed{\rho = \frac{ds}{d\psi}}$$

Theorem ②:- To find the formulae of radius of curvature in cartesian form or establish the formulae ~~$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{}$~~

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \quad \text{or} \quad \rho = \frac{\left[1+y_1^2\right]^{3/2}}{y_2}$$

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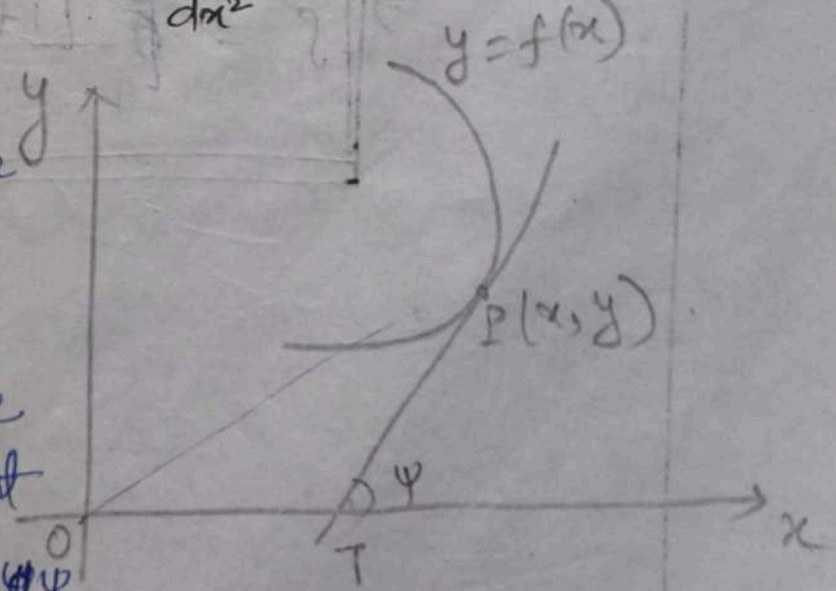
Proof:-

Let the equ of the curve $y = f(x)$.

Let $P(x, y)$ be any point on the curve & let the tangent at P makes an angle ψ with x-axis.



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Thus, we know that $\frac{dy}{dx} = \tan\psi$ or $y_1 = \tan\psi$

Diff. both sides w.r.t. x, we get

$$y_2 = \sec^2\psi \cdot \frac{d\psi}{dx}$$

$$\Rightarrow y_2 = \sec^2\psi \cdot \frac{d\psi}{ds} \cdot \frac{ds}{dx}$$

$\because \tan\psi = \frac{dy}{dx}$

$\frac{\sin\psi}{\cos\psi} = \frac{dy/ds}{dx/ds}$

$\therefore \sin\psi = \frac{dy}{ds}$

& $\cos\psi = \frac{dx}{ds}$

$\rho = \frac{ds}{d\psi} \Rightarrow \frac{1}{\rho} =$

From equ (i),

$$y_2 = \sec^2\psi \cdot \times \frac{1}{\rho} \times \frac{1}{\cos\psi}$$

$$\Rightarrow y_2 = \frac{1}{\rho}\sec^3\psi$$

$$\Rightarrow \boxed{\rho = \frac{\sec^3\psi}{y_2}} \Rightarrow \rho = \frac{(\sec^2\psi)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{(1+\tan^2\psi)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{[1+y_1^2]^{3/2}}{y_2} \quad [\because y_1 = \tan\psi]$$

$$\Rightarrow \boxed{\rho = \frac{[1+(\frac{dy}{dx})^2]^{3/2}}{d^2y/dx^2}}$$ proved

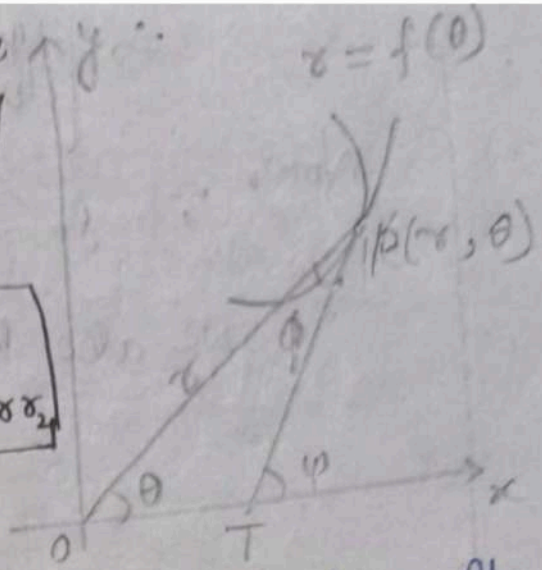
Find the radius of curvature in polar form [(r, θ) form]

OR

Establish the formula

$$\rho = \frac{[r^2 + r_1^2]^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

where, $r_1 = \frac{dr}{d\theta}$.



Proof:- Let $P(r, \theta)$ be any point on the curve, then $(OP = r)$, $\angle POX = \theta$.

We know that,

$$\tan\phi = r\frac{d\theta}{dr} = r\frac{1}{dr/d\theta}$$

$$\boxed{\tan\phi = \frac{r}{r_1}}$$

$$\Rightarrow \boxed{\phi = \tan^{-1}\left(\frac{r}{r_1}\right)}$$

Now, $\because$ PT is the tangent at P, which makes an angle ψ with x-axis, then from fig.

$$\boxed{\psi = \theta + \phi} \Rightarrow \boxed{\psi = \theta + \tan^{-1}\left(\frac{r}{r_1}\right)}$$

Diff. both sides w.r.t. θ, we get.

$$\frac{d\psi}{d\theta} = 1 + \frac{1}{1+\left(\frac{r}{r_1}\right)^2} \times \frac{(r_1 \cdot r_1) - (r r_2)}{r_1^2}$$

$$= 1 + \frac{r_1^2}{(r_1^2 + r^2)} \times \frac{(r_1^2) - (r r_2)}{r_1^2}$$

$$= \frac{(r_1^2 + r^2) + r_1^2 - r r_2}{(r_1^2 + r^2)}$$

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$$\Rightarrow \boxed{\frac{d\psi}{d\theta} = \frac{r^2 + 2r_1^2 - rr_2}{(r_1^2 + r_1^2)}}$$

Now, $\because \rho = \frac{ds}{d\psi}$ $\Rightarrow \boxed{\rho = \frac{\frac{ds}{d\theta}}{\frac{d\psi}{d\theta}}}$

We also know that, in any curve,

$$\boxed{\frac{ds}{d\theta} = \sqrt{r^2 + r_1^2}}$$

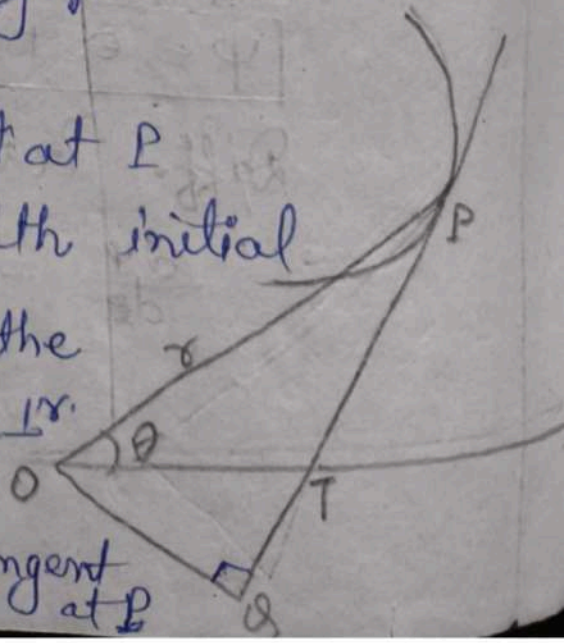
$$\Rightarrow \rho = \frac{\sqrt{r^2 + r_1^2}}{\left[\frac{r^2 + 2r_1^2 - rr_2}{(r^2 + r_1^2)}\right]}$$

$$\Rightarrow \boxed{\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}}$$ proved

Q To find the radius of curvature in pedal f
or establish the formula $\boxed{\rho = r\frac{dr}{dp}}$

⇒ Proof:- Let $p(r, \theta)$ be any point on the curve.

Let PT is the tangent at P makes any angle ψ with initial line OX. OQ = p be the length of $\perp^r$ from 'O' on the tangent at P



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Let $\angle OPT = \phi$, $\angle POT = \theta$, then,

We know that, $\boxed{\tan\phi = \frac{r\,d\theta}{dr}}$

$\Rightarrow \sin\phi = r\,d\theta/ds.$

$\cos\phi = dr/ds.$

Now, from fig.

In right angled $\triangle OPQ$, $\sin\phi = \frac{OQ}{OP} = \frac{p}{r}$

$\therefore p = r\sin\phi$ — (1)

Diff - w.r.t. r, we get

$$\frac{dp}{dr} = 1.\sin\phi + r\cos\phi \cdot \frac{d\phi}{dr}$$

$$= r\frac{d\theta}{ds} + r.\frac{dr}{ds}\cdot\frac{d\phi}{ds}$$

$$= r\frac{d}{ds}(\theta+\phi)$$

$$\boxed{\frac{dp}{dr} = r\frac{d}{ds}(\theta+\phi) = r\frac{d\psi}{ds}.}$$

Since, $\boxed{\psi = \theta+\phi}$ from fig.

$\Rightarrow \frac{dp}{dr} = r.\frac{1}{\rho}$ $\Rightarrow \boxed{\rho = \frac{r\,dr}{dp}}$

$\because \rho = \frac{ds}{d\psi}$

$\therefore \frac{1}{\rho} = \frac{d\psi}{ds}$

Proved

Q Theorem:— To find radius of curvature in tangential polar (p, ψ) form

or

establish the formula $\boxed{\rho = p + \frac{d^2p}{d\psi}}$

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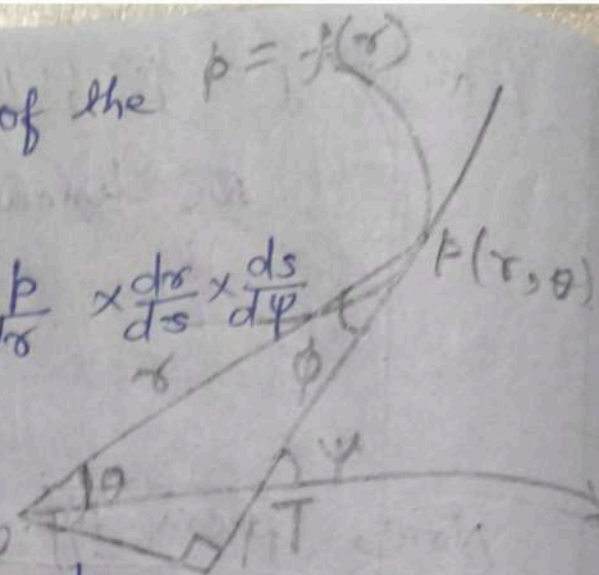
Proof :- Let the equ of the curve be $p = f(\psi)$.



We have, $\frac{dp}{d\psi} = \frac{dp}{dr} \times \frac{dr}{ds} \times \frac{ds}{d\psi}$

$\Rightarrow \frac{dp}{d\psi} = \frac{dp}{dr} \cdot \frac{dr}{ds} \cdot \rho$

$\Rightarrow \frac{dp}{d\psi} = \frac{dp}{dr} \cdot \frac{dr}{ds} \times \frac{r\,dr}{dp}$

$\Rightarrow \frac{dp}{d\psi} = \frac{r\,dr}{ds} \Rightarrow \frac{dp}{d\psi} = r\cos\phi$ [$\because \cos$ =

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Then, again from fig, we also know that, $\sin\phi = \frac{OQ}{OP} = \frac{p}{r}$

$\Rightarrow \boxed{p = r\sin\phi}$ —(2)

Squaring & adding equ (1) & (2), we get

$$\boxed{p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2}$$

Diff both sides w.r.t. p, we get

$$2p + 2\frac{dp}{d\psi} \times \frac{d^2p}{dp\,d\psi^2} = 2r\frac{dr}{dp}$$

$$\Rightarrow p + \frac{dp}{d\psi} \cdot \frac{d}{d\psi}\left(\frac{dp}{d\psi}\right) \cdot \frac{d\psi}{dp} = \frac{r\,dr}{dp}$$

$\left[\because \rho = \frac{r\,dr}{dp}\right]$ $\Rightarrow p + \frac{d^2p}{d\psi^2} = \rho \Rightarrow \boxed{\rho = p + \frac{d^2p}{d\psi^2}}$

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Find the radius of curvature for the catenary $y = c\cosh\left(\frac{x}{c}\right)$.

Sol^{n}:- The equation of catenary is

$$y = c\cosh\frac{x}{c} \quad \text{---} \quad (i)$$

Diff (i) w.r.t. x, we get.

$$\frac{dy}{dx} = c\sinh\frac{x}{c}\cdot\frac{1}{c} = \sinh\frac{x}{c}$$

$$\frac{dy}{dx} = \sinh\frac{x}{c} \quad \text{---} \quad (ii)$$

Diff equ (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \cosh\frac{x}{c}\cdot\frac{1}{c}$$

We have $\rho = \dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$.

$$\rho = \frac{\left(1+\sinh^2\frac{x}{c}\right)^{3/2}}{\frac{1}{c}\cosh\frac{x}{c}} = \frac{c\left(\cosh^2\frac{x}{c}\right)^{3/2}}{\cosh\frac{x}{c}} = \frac{c\cosh^3\frac{x}{c}}{\cosh\frac{x}{c}}$$

$\Rightarrow \rho = c\cosh^2\frac{x}{c}$

$\Rightarrow \rho = c\cdot\frac{y^2}{c^2}$

$\Rightarrow \boxed{\rho = \frac{y^2}{c}}$ Ans

[from (i), $\frac{y}{c} = \cosh\frac{x}{c}$. $\left(\frac{y}{c}\right)^2 = \cosh^2\frac{x}{c}$]

Q8. Find the radius of curvature at the point (x, y) on the parabola $y^2 = 4ax$.

Sol^{n}:- The curve is $y^2 = 4ax$ — (i).

Diff. both sides w.r.t. x.

$$2y\frac{dy}{dx} = 4a\cdot 1.$$

$$\frac{dy}{dx} = \frac{2a}{y} \Rightarrow \frac{dy}{dx} = \frac{2a}{y} \quad \text{---} \quad (ii)$$

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Again diff. both sides w.r.t. 'x'.

$$\frac{d^2y}{dx^2} = 2a\left(-\frac{1}{y^2}\right)\cdot\frac{dy}{dx} = -\frac{2a}{y^2}\cdot\frac{2a}{y} \quad \text{(from (3))}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{4a^2}{y^3}$$

Now, from formula

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\left(\frac{2a}{y}\right)^2\right]^{3/2}}{\frac{4a^2}{y^3}}$$

$$= \frac{\left[1+\frac{4a^2}{y^2}\right]^{3/2}}{\frac{4a^2}{y^3}}$$

[in magnitude neglecting -ve sign because ρ is length radius]

$$= \frac{\left[\frac{y^2+4a^2}{y^2}\right]^{3/2}}{\frac{4a^2}{y^3}}$$

$$= \frac{\frac{(y^2+4a^2)^{3/2}}{y^3}}{\frac{4a^2}{y^3}} = \frac{(y^2+4a^2)^{3/2}}{4a^2}$$

$$= \frac{(4ax+4a^2)^{3/2}}{4a^2} \quad [\text{from equ (i)}] \quad [\because y^2 = 4ax]$$

$$= \frac{[4a(x+a)]^{3/2}}{4a^2} = \frac{(4a)^{3/2}(x+a)^{3/2}}{4a^2}$$

$$= \frac{4^{3/2}\cdot a^{3/2}(x+a)^{3/2}}{4a^2} = \frac{4^{\frac{3}{2}-1}(x+a)^{3/2}}{a^{2-\frac{3}{2}}}$$

$$= \frac{4^{\frac{1}{2}}(x+a)^{3/2}}{a^{\frac{1}{2}}} = \frac{2(x+a)^{3/2}}{\sqrt{a}}$$

which is the required results. Ans

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Find the radius of curvature of the curve,

$y^2 = 8x$.

Soln:- We have $y^2 = 8x$. — (i)

Diff. both sides w.r.t. x, we get

$2y \cdot \frac{dy}{dx} = 8 \Rightarrow \frac{dy}{dx} = \frac{4}{y}$ — (ii)

Again diff. w.r.t. x,

$\frac{d^2y}{dx^2} = -\frac{4}{y^2} \cdot \frac{dy}{dx} = -\frac{4}{y^2} \cdot \frac{4}{y}$ [using equ (ii)]

$\Rightarrow \boxed{\frac{d^2y}{dx^2} = -\frac{16}{y^3}}$

From formula, $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\frac{16}{y^2}\right)^{3/2}}{y_2 = \left|-\frac{16}{y^3}\right|}$

$\Rightarrow \rho = \frac{1+}{}$ Now, (in magnitude, neglecting -ve sign)

$\rho = \frac{y^3}{16} \frac{(y^2+16)^{3/2}}{y^3} = \frac{(8x+16)^{3/2}}{16}$ [using (i)]

$\Rightarrow \rho = \frac{8^{\frac{3}{2}}(x+2)^{3/2}}{16}$ $= 2^{3x+\frac{3}{2}}$ Ans

4.9 Find the radius of curvature of the curve.

$y = e^x$ & $\rho = \frac{(1+e^{2x})^{3/2}}{e^x}$

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9. If ρ be the radius of curvature of a parabola at a point whose distance measured along the curve from a fixed point is s, prove that

$$3\rho \frac{d^2\rho}{ds^2} - \left(\frac{d\rho}{ds}\right)^2 - 9 = 0.$$

Soln:- Let the equation of parabola be

$$y^2 = 4ax \quad \text{or} \quad y = 2\sqrt{a}\sqrt{x} \qquad \text{---(i)}$$

Diff. successively w.r.t. x, we get

$$y_1 = 2\sqrt{a}\cdot\frac{1}{2\sqrt{x}} = \frac{\sqrt{a}}{\sqrt{x}}.$$

$$\Rightarrow y_1 = \sqrt{a}\,x^{1/2}.$$

$$\Rightarrow y_2 = \sqrt{a}\left(-\frac{1}{2}\right)x^{-3/2} \Rightarrow y_2 = -\frac{\sqrt{a}}{2x^{3/2}}.$$

Now, from formula, ~~$\rho = \frac{1+y_1^2}{}$~~

$$\rho = \frac{}{y_2} = \frac{}{-\frac{1}{2}\frac{\sqrt{a}}{x^{3/2}}}$$

$$\Rightarrow \rho = \frac{(1+y_1^2)^{3/2}}{-\frac{1}{2}\sqrt{a}} \times x^{3/2} = \frac{(x+a)^{3/2} \cdot x^{3/2}}{-\frac{\sqrt{a}}{2} \cdot x^{3/2}}$$

$$\Rightarrow \rho = +\frac{2}{\sqrt{a}}(x+a)^{3/2}, \quad \text{(in magnitude neglecting -ve sign)}$$

Diff. both sides w.r.t. x, we get

$$\frac{d\rho}{dx} = \frac{2}{\sqrt{a}} \times \frac{3}{2}(x+a)^{\frac{1}{2}} = \frac{3}{\sqrt{a}}(x+a)^{\frac{1}{2}}.$$

We have, $\frac{ds}{dx} = \sqrt{1+y_1^2} = \sqrt{1+\frac{a}{x}} = \sqrt{\frac{x+a}{x}}$

$$\left[\because \frac{ds^2}{dx^2} = \frac{dx^2}{dx^2} + \frac{dy^2}{dx^2}\right]$$

$$\left[\because y_1 = \sqrt{\frac{a}{x}}\right]$$

$$\Rightarrow \boxed{\frac{dx}{ds} = \sqrt{\frac{x}{x+a}}}$$

Now, $\frac{d\rho}{ds} = \frac{d\rho}{dx} \cdot \frac{dx}{ds} = \frac{3}{\sqrt{a}}\sqrt{x+a} \cdot \frac{\sqrt{x}}{\sqrt{x+a}} = \frac{3}{\sqrt{a}}\sqrt{x}$

$$\Rightarrow \boxed{\frac{d\rho}{ds} = \frac{3}{\sqrt{a}}\sqrt{x}} \Rightarrow \frac{d^2\rho}{ds^2} = \frac{d}{ds}\left(\frac{3}{\sqrt{a}}\sqrt{x}\right) = \frac{d}{dx}\left(\frac{3}{\sqrt{a}}\sqrt{x}\right)\frac{dx}{ds}$$

$$\Rightarrow \frac{d^2\rho}{ds^2} = \frac{3}{\sqrt{a}} \cdot \frac{1}{2\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x+a}} = \frac{3}{2\sqrt{a}\sqrt{x+a}}$$

$$\Rightarrow \frac{d^2\rho}{ds^2} = \frac{3}{2\sqrt{a(x+a)}}$$

L.H.S. $= 3\rho\frac{d^2\rho}{ds^2} - \left(\frac{d\rho}{ds}\right)^2 - 9.$

$$= 3 \times \frac{2}{\sqrt{a}}(x+a)^{3/2} \times \frac{3}{2\sqrt{a}\sqrt{x+a}} - \left[\frac{3}{\sqrt{a}}\sqrt{x}\right]^2 - 9.$$

$$= \frac{9(x+a)}{a} - \frac{9x}{a} - 9 = \frac{9(x+a) - 9x - 9a}{a}.$$

$\Rightarrow$ L.H.S. = R.H.S. Proved $\left[\frac{d\rho}{ds} = 0\right]$

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Q. Find the radius of curvature at any point (r, θ) for the curve $r^n = a^n \cos n\theta$.

Sol^{n}:- We have,

$$r^n = a^n \cos n\theta.$$

Diff. both sides w.r.t., θ, we have,

$$n r^{n-1} \cdot \frac{dr}{d\theta} = -a^n \sin n\theta \times n.$$

$$\Rightarrow r^{n-1} \cdot r_1 = -a^n \sin n\theta \quad \text{---(i)}$$

$$\Rightarrow \frac{r^n}{r} \cdot r_1 = -a^n \sin n\theta.$$

$$\Rightarrow \frac{a^n \cos n\theta}{r} r_1 = -a^n \cos n\theta \quad [\text{from given}]$$

$$\Rightarrow r_1 = -\frac{r \sin n\theta}{\cos n\theta}.$$

$$\therefore r_1 = -r \tan n\theta \quad \text{---(ii)}$$

Again diff. (i) both sides w.r.t. θ, we get

$$[(n-1) r^{n-2} \cdot r_1] \times r_1 + r^{n-1} \cdot r_2 = -a^n \cos n\theta \cdot n.$$

$$\Rightarrow (n-1) \frac{r^n}{r^2} \cdot r_1^2 + \frac{r^n}{r} \cdot r_2 = -a^n \cos n\theta \cdot n.$$

$$\Rightarrow r^n \left[\frac{(n-1) r_1^2}{r^2} + \frac{r_2}{r}\right] = -r^n \cdot n \quad [\text{from question}]$$

$$\Rightarrow \frac{(n-1) r_1^2 + r r_2}{r^2} = -n$$

$$\Rightarrow (n-1) r_1^2 + r r_2 = -n r^2$$

$$\Rightarrow r r_2 = -r^2 n - r_1^2 (n-1) \quad \text{---(iii)}$$

Putting the value of (r_1) in (iii) from (ii), we get

$$r r_2 = -r^2 n - r^2 \tan^2 n\theta (n-1).$$

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$= -r^2 n - r^2 n \tan^2 n\theta + r^2 \tan^2 n\theta$.

$= -r^2 n (1 + \tan^2 n\theta) + r^2 \tan^2 n\theta$.

$\Rightarrow r r_2 = -r^2 n \sec^2 n\theta + r^2 \tan^2 n\theta$.

Now, we know that,

$$\boxed{\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}} \quad \text{---} \quad (iv)$$

So, $r^2 + r_1^2 = r^2 + r^2 \tan^2 n\theta = r^2(1 + \tan^2 n\theta)$

$$\Rightarrow \boxed{r^2 + r_1^2 = r^2 \sec^2 n\theta} \quad \text{---} \quad (v)$$

Also, $r^2 + 2r_1^2 - r r_2 = r^2 + 2r^2 \tan^2 n\theta + r^2 n \sec^2 n\theta - r^2 \tan^2 n\theta$.

$= r^2 + r^2 n \sec^2 n\theta + r^2 \tan^2 n\theta$

$= r^2 + r^2 n \sec^2 n\theta + r^2 (\sec^2 n\theta - 1)$

$= r^2 + r^2 n \sec^2 n\theta + r^2 \sec^2 n\theta - r^2$.

$$\Rightarrow r^2 + 2r_1^2 - r r_2 = r^2 \sec^2 n\theta (n+1) \quad \text{---} \quad (vi)$$

Putting the values of (v) & (vi) in above eq^n (iv), we get

$$\rho = \frac{(r^2 \sec^2 n\theta)^{3/2}}{r^2 \sec^2 n\theta (n+1)} = \frac{r^{2\times\frac{3}{2}} \sec^{2\times\frac{3}{2}} n\theta}{r^2 \sec^2 n\theta (n+1)}$$

$$\Rightarrow \rho = \frac{r^3 \sec^3 n\theta}{r^2 \sec^2 n\theta (n+1)} = \frac{r}{(n+1)} \cdot \frac{1}{\cos n\theta} = \frac{r}{(n+1)} \cdot \frac{1}{\frac{r^n}{a^n}}$$

$$[\because r^n = a^n \cos n\theta]$$

$$\Rightarrow \boxed{\rho = \frac{a^n}{(n+1) r^{n-1}}}$$ Ans

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Q) Find radius of curvature for the curve

$$r = a(1-\cos\theta).$$

Hence, deduce that ρ^2/r is constant.

Sol^n:- We have, $r = a(1-\cos\theta)$

Diff. both sides w.r.t. θ

$$r_1 = a(0+\sin\theta).$$

$$\therefore r_1 = a\sin\theta.$$

Again diff. both sides w.r.t. θ, we get

$$r_2 = a\cos\theta.$$

Now,

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2} = \frac{\{(a-a\cos\theta)^2 + a^2\sin^2\theta\}^{3/2}}{(a-a\cos\theta)^2 + 2a^2\sin\theta - a(1-\cos\theta)a\cos\theta}$$

$$= \frac{[a^2 - 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta]^{3/2}}{a^2 - 2a^2\cos\theta + a^2\cos^2\theta + 2a^2\sin\theta - a^2\cos\theta + a^2\cos\theta}$$

$$= \frac{[a^2 - 2a^2\cos\theta + a^2(\cos^2\theta + \sin^2\theta)]^{3/2}}{a^2 + 2a^2\sin^2\theta + 2a^2\cos^2\theta - 3a^2\cos\theta}.$$

$$= \frac{[a^2 - 2a^2\cos\theta + a^2]^{3/2}}{a^2 + 2a^2(\sin^2\theta + \cos^2\theta) - 3a^2\cos\theta}$$

$$= \frac{[2a^2 - 2a^2\cos\theta]^{3/2}}{a^2 + 2a^2 - 3a^2\cos\theta} = \frac{[2a^2(1-\cos\theta)]^{3/2}}{3a^2(1-\cos\theta)}$$

$$= \frac{(2a^2)^{3/2}\cdot(1-\cos\theta)^{3/2}}{3a^2(1-\cos\theta)} = \frac{2^{3/2}\cdot a^3}{3a^2}(1-\cos\theta)^{1/2}$$

$$= \frac{2}{3}\sqrt{2}\,a\,(1-\cos\theta)^{1/2}$$

$$= \frac{2}{3}\sqrt{a^2 2(1-\cos\theta)}.$$

$$\Rightarrow \rho = \frac{2}{3}\sqrt{2a\cdot a\cdot 2(1-\cos\theta)}$$

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$\Rightarrow \rho = \frac{2}{3}\sqrt{2ar}$.

Now, squaring both sides.

$\rho^2 = \frac{4}{9} \cdot 2ar$.

or $\frac{\rho^2}{r} = \frac{4}{9} \cdot 2a = \frac{8a}{9}$ (constant).

$\Rightarrow \boxed{\frac{\rho^2}{r} = \text{constant}}$ ϕ Ans

Q. For any curve, prove the formula, $\rho = \frac{r}{\sin\phi\left(1 + \frac{d\phi}{d\theta}\right)}$, where $\tan\phi = r\frac{d\theta}{dr}$.

Proof :- We have,

$\tan\phi = r\frac{d\theta}{dr} = r \cdot \frac{1}{\frac{dr}{d\theta}} = \frac{r}{r_1}$.

$\therefore r_1 = \frac{r_e}{\tan\phi} = r\cot\phi$ ——— (i)

$\therefore \boxed{\frac{r_1}{r} = \cot\phi}$ ——— (i)

Diff. both sides w.r.t. θ, we get.

$\left[\frac{r \cdot r_2 - r_1^2}{r^2} = -\operatorname{cosec}^2\phi \cdot \frac{d\phi}{d\theta}\right] \times (-1)$

$\Rightarrow r_1^2 - rr_2 = r^2\operatorname{cosec}^2\phi \frac{d\phi}{d\theta}$

Adding r_1^2 both sides, we get

$2r_1^2 - rr_2 = r^2\operatorname{cosec}^2\phi \frac{d\phi}{d\theta} + r_1^2$.

$\Rightarrow 2r_1^2 - rr_2 = r^2\operatorname{cosec}^2\phi \frac{d\phi}{d\theta} + r^2\cot^2\phi$.

$= r^2\operatorname{cosec}^2\phi \frac{d\phi}{d\theta} + r^2(\operatorname{cosec}^2\phi - 1)$.

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$\Rightarrow r^2 + 2r_1^2 - rr_2 = r^2 \operatorname{cosec}^2\phi \frac{d\phi}{d\theta} + r^2$

$- r^2 + r^2$

$\Rightarrow r^2 + 2r_1^2 - rr_2 = r^2 \operatorname{cosec}^2\phi \left(1 + \frac{d\phi}{d\theta}\right)$

Now, $\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$

$\Rightarrow \rho = \frac{(r^2 + r^2\cot^2\phi)^{3/2}}{r^2 \operatorname{cosec}^2\phi \left(1 + \frac{d\phi}{d\theta}\right)}$ [fro

$\Rightarrow \rho = \frac{(r^2 + r^2\cot^2\phi)^{3/2}}{r^2 \operatorname{cosec}^2\phi}$

$\Rightarrow \rho = \frac{r^3(1 + \cot^2\phi)^{3/2}}{r^2 \operatorname{cosec}^2\phi \left(1 + \frac{d\phi}{d\theta}\right)}$

$\Rightarrow \rho = \frac{r(\operatorname{cosec}^2\phi)^{3/2}}{\operatorname{cosec}^2\phi \left(1 + \frac{d\phi}{d\theta}\right)}$

$\Rightarrow \rho = \frac{r \operatorname{cosec}^3\phi}{\operatorname{cosec}^2\phi \left(1 + \frac{d\phi}{d\theta}\right)} = \frac{r \operatorname{cosec}}{\sin(1+}$

$\Rightarrow \boxed{\rho = \frac{r}{\sin\phi \left(1 + \frac{d\phi}{d\theta}\right)}}$ proved

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Date 07/11/23

Homogeneous Function

$a x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + a_3 x^{n-3} y^3 + \cdots + a_n y^n$

(is a function of x & y of degree 'n'.

(Here, every term is of degree n), which

Such a function is called homogeneous function of degree 'n'. Also,

$$f(x,y) = u(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + a_3 x^{n-3} y^3 + \cdots + a_n y^n$$

$$= x^n \left[a_0 + a_1 \frac{y}{x} + a_2 \left(\frac{y}{x}\right)^2 + a_3 \left(\frac{y}{x}\right)^3 + \cdots + a_n \left(\frac{y}{x}\right)^n \right]$$

$$\Rightarrow \boxed{u(x,y) = x^n \phi\left(\frac{y}{x}\right)}$$, Here, every homogeneous fn of x & y of degree 'n' & can be written as.

$$\boxed{u(x,y) = x^n \phi\left(\frac{y}{x}\right)}$$

Euler's Theorem :-

State & prove Euler's theorem

Statement :- If u be a homogeneous function of x & y of degree (n). Then $\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}}$

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Proof :- Since, u is a homogeneous function of x & y of degree (n). Then $\boxed{u = x^n \phi\left(\frac{y}{x}\right)}$ — (1)

Diff. equo (1) partially w.r.t. x, we get

$$\boxed{\frac{\partial u}{\partial x} = nx^{n-1}\phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right)}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = nx^{n-1}\phi\left(\frac{y}{x}\right) - x^n \phi'\left(\frac{y}{x}\right)\cdot\frac{y}{x^2}}$$

Both sides, on multiplying by x, we get

$$\boxed{x\frac{\partial u}{\partial x} = nx^n \phi\left(\frac{y}{x}\right) - x^{n-1}\phi'\left(\frac{y}{x}\right)\cdot\frac{y}{x}} \quad — (2)$$

Again diff. ~~both~~ equo (i) partially w.r.t. y, we get

$$\boxed{\frac{\partial u}{\partial y} = 0 + x^n \phi'\left(\frac{y}{x}\right)\cdot\frac{1}{x}}$$

Multiplying both sides by y, we get

$$\boxed{y\frac{\partial u}{\partial y} = x^n \phi'\left(\frac{y}{x}\right)\cdot\frac{y}{x}} \quad — (3)$$

Now, from (2) + (3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n\phi\left(\frac{y}{x}\right) - x^{n-1}\phi'\left(\frac{y}{x}\right)\cdot\frac{y}{x} + x^n\phi'\left(\frac{y}{x}\right)\cdot\frac{y}{x}$$

Now, using equo (1),

$$\Rightarrow \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$$

proved

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Theorem :-

If u be a homogeneous fn of x & y of degree 'n' then prove that $\boxed{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.}$

Proof :- Since, u is a homogeneous fn of x & y of degree 'n'. Then by Euler's theorem,

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu} \quad \text{---} (1)$$

Diff. equo (1) partially w.r.t. x, we get

$$\boxed{\left(1\cdot\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2}\right) + \left(0 + y\frac{\partial^2 u}{\partial x \partial y}\right) = n\frac{\partial u}{\partial x}.}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x}.$$

Multiplying both sides by x, we get.

$$\boxed{x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = x(n-1)\frac{\partial u}{\partial x}.} \quad \text{---} (2)$$

Again diff. equo (1) partially w.r.t. y, we get

$$x\frac{\partial^2 u}{\partial y \partial x} + y\left(1\cdot\frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2}\right) = n\cdot\frac{\partial u}{\partial y}.$$

Multiplying $\boxed{x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}}$

Multiplying both sides by 'y', we get

$$\boxed{xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = y(n-1)\frac{\partial u}{\partial y}} \quad \text{---}$$

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Adding equ (2) & (3), we get

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = (n-1)\left(x\frac{\partial u}{\partial x}\right.$$

Using equ (1), we get

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = n(n-1)u$$

Prob

If $v = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$. Then show that

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + \frac{1}{2}\cot v = 0$$

Proof:- We have, $v = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$.

$$\therefore \quad v = \cos^{-1} z \quad \text{[where } z = \text{]}$$

$$\Rightarrow \quad z = \cos v \quad \text{---} (1)$$

Now, since ~~$z = \cos v$~~ $z = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

$$\Rightarrow z = \frac{x\left[1+\frac{y}{x}\right]}{\sqrt{x}\left[1+\sqrt{\frac{y}{x}}\right]} =$$

$$\text{i.e. } \Rightarrow z = x^{\frac{1}{2}}\phi\left(\frac{y}{x}\right).$$

Hence, z is a homogeneous

Then, by Euler's theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

$$\therefore x\frac{\partial(\cos v)}{\partial x} + y\frac{\partial(\cos v)}{\partial y} = \frac{n}{2}\cos v \quad \text{— (2)}$$

$$\Rightarrow x(-\sin v)\frac{\partial v}{\partial x} + y(-\sin v)\frac{\partial v}{\partial y} = \frac{1}{2}\cos v$$

$$\Rightarrow (\sin v)\left[x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right] = \frac{1}{2}\cos v$$

$$\Rightarrow x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = -\frac{1}{2}\cot v$$

$$\Rightarrow \boxed{x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + \frac{1}{2}\cot v = 0}$$ proved

Q. If $u = \sin^{-1}\frac{x^2+y^2}{x+y}$, show that $\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u = 0}$

Soln:- ∵ We have $u =$

Q. If $u = \sin^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{x}{y}\right)$. Then show that $\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0}$

Soln:- ∵ We have $\boxed{u = \sin^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{x}{y}\right).}$ — (1)

Diff. eqn (1) partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \cdot \left(\frac{-y}{x^2}\right) + \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(\frac{1}{y}\right)$$

⇒ Multiplying both sides by x, we get

$$x\frac{\partial u}{\partial x} = \frac{x \cdot x^2}{\sqrt{(x^2-y^2)}} \left(\frac{-y}{x^2}\right)$$

⇒ **Soln:-** Equo (i) can be written as

$$u = x^0\left[\sin^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{x}{y}\right)\right].$$

$$\Rightarrow u = x^0 \phi\left(\frac{y}{x}\right).$$

∴ Again Diff. both sides partially w.r.t x, we get

$$\frac{\partial u}{\partial x} = x\phi'\left(\frac{y}{x}\right).\left(-\frac{1}{x^2}\right) + x$$

Q. If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, then prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1-4\sin^2 u)\sin 2u$$

⇒ **Proof:-** Since, we have, $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$\Rightarrow u = \tan^{-1}(z) \quad [\text{where } z = \ldots$$

$$\Rightarrow z = \tan u$$

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Maxima & Minima for two variables

(Diff. Calculus of 2 or more Variables)

(1) Working Rule :-

Step 1: Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$

Step 2: Solving the equations

$\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$.

Simultaneously for x & y

Let (x_1, y_1), (x_2, y_2), ---- be the pairs of the values (of ----)

Step 3:- Find $A = \frac{\partial^2 f}{\partial x^2}$, $B = \frac{\partial^2 f}{\partial x \partial y}$, $C = \frac{\partial^2 f}{\partial y^2}$ and calculate the values of A, B & C for pairs of values in step 2.

Step 4: (i) If $AC - B^2 > 0$ & $A < 0$ for a pair of values say (x_1, y_1), then $f(x, y)$ maximum for this pair & $f(x_1, y_1)$ is the maximum value of $f(x, y)$.

(ii) If $AC - B^2 > 0$ and $A > 0$ for a pair of values say (x_2, y_2), then $f(x, y)$ is minimum for this pair. and $f(x_2, y_2)$ is the minimum value of $f(x, y)$.

(iii) If $AC - B^2 < 0$ for the pair of values say (x_3, y_3) then there is neither maximum nor minimum, such points (x_3, y_3) is called saddle point.

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(iv) If $AC - B^2 = 0$ for some pair of values (a, b), the doubtful and for further investigation.

Test the function $f(x,y) = x^2 - xy + y^2 + 3x - 2y + 1$ for extreme values.

Soln:- Let $f(x,y) = x^2 - xy + y^2 + 3x - 2y + 1$.

Step 1:- (1) $\therefore \frac{\partial f}{\partial x} = 2x - y + 3$

& $\frac{\partial f}{\partial y} = -x + 2y - 2$

Step 2:- $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$.

$\Rightarrow \frac{\partial f}{\partial x} = 2x - y + 3 = 0$, i.e., $2x - y + 3 = 0$ — (i)

& $\frac{\partial f}{\partial y} = -x + 2y - 2 = 0$ — (ii)

From equ (ii) $\times 2$, we get

$-2x + 4y - 4 = 0$ — (iii)

(i) + (iii), we get, $2x - y + 3 = 0$

$-2x + 4y - 4 = 0$

$3y - 1 = 0 \Rightarrow y = \frac{1}{3}$.

$\therefore \frac{\partial f}{\partial y} = -x + 2y - 2 = 0$.

From equ (i), $2x - \frac{1}{3} + 3 = 0 \Rightarrow 2x + \frac{8}{3} = 0$

$\Rightarrow 2x = -\frac{8}{3} \Rightarrow x = -\frac{4}{3}$

Hence, the required point is $(-\frac{4}{3}, \frac{1}{3})$.

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Step 3 :- $\because \frac{\partial f}{\partial x} = 2x - y + 3 = 0$

$\therefore \boxed{A = \frac{\partial^2 f}{\partial x^2} = 2}$

$\Rightarrow$ & $\frac{\partial f}{\partial y} = -x + 2y - 2 = 0$

$\therefore \because a = \frac{\partial^2 f}{\partial y^2} = 2 \Rightarrow \boxed{\frac{\partial^2 f}{\partial x \partial y} = -1}$

Step 4 :- $AC - B^2 = 2 \times 2 - (-1)^2 = 4 - 1 = 3 > 0$

i.e., $\boxed{AC - B^2 > 0}$ & $\boxed{A = 2 < 0}$

Therefore $f(x,y)$ has minimum value at $(\frac{-1}{3}$

Hence, the minimum value is

$f(-\frac{1}{3}, \frac{1}{3}) = (-\frac{1}{3})^2 - (-\frac{1}{3}) \times \frac{1}{3} + (\frac{1}{3})^2 + 3(-\frac{1}{3})$

$\Rightarrow \boxed{f(-\frac{1}{3}, \frac{1}{3}) = \frac{16}{9} + \frac{8}{9} + \frac{1}{9} + \frac{4}{9} - \frac{2}{3} + 1}$

Find all the maximum & minimum va of the function is defined by

$f(x,y) = x^3 + y^3 - 63(x+y) + 12(xy)$

Sol^n :- Let $f(x,y) = x^3 + y^3 - 63(x+y) + 12xy$

Step I :- $\frac{\partial f}{\partial x} = 3x^2 - 63 + 12y$

& $\frac{\partial f}{\partial y} = 3y^2 - 63 + 12x$

Step II :- $\because \frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$\therefore 3x^2 - 63 + 12y = 0$ & $3y^2 - 63 + 12x = 0$

$\Rightarrow x^2 + 4y - 21 = 0$ &

From equ (i) - (ii)

$$x^2 - y^2 + 4(y-x) = 0$$

$$\Rightarrow (x-y)(x+y) - 4(x-y) = 0$$

$$\Rightarrow (x-y)(x+y-4) = 0$$

either $x-y=0$ or $x+y-4=0$

$\Rightarrow x = y$ or $x+y=4 \Rightarrow x = 4-y$

Putting $(x=y)$ in equ (ii), we get

$$y^2 + 4y - 21 = 0$$

$\Rightarrow$ Here, $a=1$, $b=4$, $c=-21$.

$$\Rightarrow y = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-4 \pm \sqrt{16+84}}{2} = -7 \& 3$$

$y = -7 \& 3$.

$\therefore x = y = -7; 3$

So, Points are $(-7,-7)$ & $(3,3)$

Again put $x = 4-y$ in above equ (ii), we have,

$$(4-y)^2 + (4y - 21) = 0$$

$$\Rightarrow 16 + y^2 - 8y + 4y - 21 = 0$$

$$\Rightarrow y^2 - 4y - 5 = 0 \Rightarrow y^2 + y - 5y - 5 = 0$$

$$\Rightarrow y(y+1) - 5(y+1) = 0$$

Here, $a=1$, $b=-4$, $c=-5$

$$\therefore y = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{4 \pm \sqrt{16-20}}{2} \Rightarrow (y-5)(y+1) = 0.$$

$\Rightarrow y = 5$ or $y = -1$

$\Rightarrow$ Points are $(-1,5)$ & $(5,-1)$.

So, stationary points are $(-7,-7)$, $(3,3)$, $(-1,5)$ or $(5,-1)$. A

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Step 3:- $A = \frac{\partial^2 f}{\partial x^2} = 6x$ & $C = \frac{\partial^2 f}{\partial y^2} = 6y$

$B = \frac{\partial^2 f}{\partial x \partial y} = 12$

$\Rightarrow AC - B^2 = 6x \times 6y - (12)^2 = 36xy - 144$

$\Rightarrow \boxed{A = 6x}$

Step 4:- $A(-7, 7)$.

$\Rightarrow AC - B^2 = 36xy - 144 = 36 \times (-7) \times (-7) = 162$

$\Rightarrow AC - B^2 > 0$.

& $A = 6x = 6 \times (-7) = -4 < 0$.

i.e., $A < 0$

Hence, $AC - B > 0$ & $A < 0$.

$\therefore$ There is a maximum value at the point $(-7,$

Thus, from equation (1),

$f(-7, -7) = (-7)^3 + (-7)^3 - 63(-7-7) + 12 \times (-7) \times (-7) =$

at the point $(3, 3)$.

$\Rightarrow AC - B^2 = 36xy - 144 = 36 \times 3 \times 3 - 144 = 180 > 0$

$\Rightarrow AC - B^2$

& $A = 6x = 6 \times 3 = 18 > 0$

i.e., $A > 0$.

$\therefore f(x, y)$ is minimum at the point $(3, 3)$

Hence, the minimum value of $f(x, y)$.

$f(3, 3) = 3^3 + 3^3 - 63(3+3) + 12 \times 3 \times 3 = -2$

At $(-1, 5)$

$AC - B^2 = 36xy - 144 < 36 \times (-1) \times (5$

$= -144 < 180$.

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$f(-1,5) = (-1)^3 + (5)^3 - 63(-1+5) + 12(-1)(5) = 144$

neither maxima nor minima. (saddle point.)

Find maxima & minima

(i) $f(x,y) = x^3 + y^3 - 3axy$.

(ii) $f(x,y) = x^3 y^2 (1-x-y)$.

(iii) $\sin x + \sin y + \sin(x+y)$

Composite Function

$z = f(x,y)$, $x = \phi(u,v)$ & $y = \psi(u,v)$

$z = f(x,y)$, $x = \phi(t)$, $y = \psi(t)$.

z is composite function of t.

z is composite function of r.

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Jacobian

Definition :- If u, v are two functions of two independent variables x & y, then the expression

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$ is called the Jacobian or functional determinant of u, v w.r.t. x, y and is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$.

Thus, $\frac{\partial(u,v)}{\partial(x,y)} = J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

Similarly, the Jacobians u, v, w w.r.t x, y, z is $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$.

Imp.

General definition :- If $y_1, y_2, y_3, \ldots, y_n$ are n functions of the variables $x_1, x_2, x_3, \ldots x_n$, then the determinant

$$\begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \frac{\partial y_n}{\partial x_3} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix},$$

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the Jacobian of $y_1, y_2, y_3, \ldots$ w.r.t. $x_1, x_2, x_3, \ldots, x_n$. and is generally denoted by $\frac{\partial(y_1, y_2, \ldots y_n)}{\partial(x_1, x_2, \ldots x_n)} = J\left(\frac{y_1 \ldots y_n}{x_1 \ldots x_n}\right)$. If there is no doubt about the independent variables then the Jacobian is also denoted by $J(y_1, y_2 \ldots y_n)$

Example ① :-

If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$.

Sol^{n} :- We have,

$x = r\cos\theta$

$\frac{\partial x}{\partial r} = 1.\cos\theta = \cos\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta$

& $y = r\sin\theta$

$\frac{\partial y}{\partial r} = 1.\sin\theta = \sin\theta$

& $\frac{\partial y}{\partial \theta} = r\cos\theta$.

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r$ Ans

Particular Case :-

If the functions $y_1, y_2, \ldots y_n$ are of the form :

$y_1 = f_1(x_1)$ | $y_2 = f_2(x_1, x_2)$

$y_3 = f(x_1, x_2, x_3)$

$y_n = f(x_1, x_2, \ldots, x_n)$.

Then, we have,

$$\frac{\partial(y_1, y_2, \ldots y_n)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & 0 & 0 & \cdots & 0 \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \frac{\partial y_n}{\partial x_3} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

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Since, $\frac{\partial y_1}{\partial x_2}, \cdots \frac{\partial y_1}{\partial x_3} = 0$, etc.

Q12) If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(r,\theta)}{\partial(x,y)}$.

⇒ Soln:- We have,

$x = r\cos\theta$, $y = r\sin\theta$, find

we can get $(r^2 = x^2 + y^2)$ & $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

$\Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} =$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$.

Also, $\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \left(-\frac{y}{x^2}\right) = \frac{-y/x^2}{\frac{x^2+y^2}{x^2}} =$

$\Rightarrow \frac{\partial \theta}{\partial x} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$.

And, $\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{\frac{1}{x}}{\frac{x^2+y^2}{x^2}} =$

$\Rightarrow \frac{\partial \theta}{\partial y} = \frac{r\cos\theta}{r^2}$, So, $\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$

Now,

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{vmatrix}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = +\frac{1}{r}\left[\cos^2\theta + \sin^2\theta\right] = \frac{1}{r}.$$

A

(13) If $u_1 = \frac{x_2x_3}{x_1}$, $u_2 = \frac{x_1x_3}{x_2}$ & $u_3 = \frac{x_1x_2}{x_3}$

Evaluate $J(u_1, u_2, u_3)$.

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Sol: We have,

$u_1 = \frac{x_2x_3}{x_1}$ | $u_2 = \frac{x_1x_3}{x_2}$ | $u_3 = \frac{x_1x_2}{x_3}$

$\frac{\partial u_1}{\partial x_1} = -\frac{x_2x_3}{x_1^2}$ | $\frac{\partial u_2}{\partial x_1} = \frac{x_3}{x_2}$ | $\frac{\partial u_3}{\partial x_1} = \frac{x_2}{x_3}$

$\frac{\partial u_1}{\partial x_2} = \frac{x_3}{x_1}$ | $\frac{\partial u_2}{\partial x_2} = -\frac{x_1x_3}{x_2^2}$ | $\frac{\partial u_3}{\partial x_2} = \frac{x_1}{x_3}$

$\frac{\partial u_1}{\partial x_3} = \frac{x_2}{x_1}$ | $\frac{\partial u_2}{\partial x_3} = \frac{x_1}{x_2}$ | $\frac{\partial u_3}{\partial x_3} = -\frac{x_1x_2}{x_3^2}$

Now, since $J(u_1, u_2, u_3) = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$ — (i)

Putting all values in (i), we get

$$J(u_1, u_2, u_3) = \begin{vmatrix} -\frac{x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_1x_3}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1x_2}{x_3^2} \end{vmatrix}$$

Multiplying ~~2nd~~ 1st, 2nd & III rd rows by x_1^2, x_2^2, x_3^2, we get

$$J(u_1, u_2, u_3) = \frac{1}{x_1^2x_2^2x_3^2} \begin{vmatrix} -x_2x_3 & x_1x_3 & x_1x_2 \\ x_2x_3 & -x_1x_3 & x_1x_2 \\ x_2x_3 & x_1x_3 & -x_1x_2 \end{vmatrix}$$

Perform $R_1 \to R_1 + R_3$ & $R_2 \to R_2 + R_3$; [or adding 3rd rows to 1st and 2nd]

$$\Rightarrow J(u_1,u_2,u_3) = \frac{1}{x_1^2x_2^2x_3^2}\begin{vmatrix} -x_2x_3 & & \\ x_2x_3 & -x_1x_3 & x_1x_2 \\ x_2x_3 & x_1x_3 & -x_1x_2 \end{vmatrix}$$

Expanding w.r.t. 1st row, we get

$$\Rightarrow J(u_1,u_2,u_3) = \frac{1}{x_1^2x_2^2x_3^2}\left[0\left(-\;-\right) - 2x_1x_3(-2x_1x_3) + 0\left(-\;-\right)\right]$$

$$= \frac{1}{x_1^2x_2^2x_3^2} \times 4x_1^2x_2^2x_3^2 = 4 \text{ Ans}$$

(4) If $x = a\cosh\alpha\cos\beta$, $y = a\sinh\alpha\sin\beta$, show that $\frac{\partial(x,y)}{\partial(\alpha,\beta)} = \frac{1}{2}a^2(\cos 2\alpha - \cos 2\beta)$.

Soln:- We have,

$$x = a\cosh\alpha\cos\beta \qquad y = a\sinh\alpha\sin\beta$$

VVI $\frac{\partial x}{\partial\alpha} = a\sinh\alpha\cos\beta$, $\frac{\partial y}{\partial\alpha} = +a\cosh\alpha\sin\beta$

$\frac{\partial x}{\partial\beta} = -a\cosh\alpha\sin\beta$, $\frac{\partial y}{\partial\beta} = a\sinh\alpha\cos\beta$.

$\because d(\cosh\alpha) = \sinh\alpha$

& $d(\sinh\alpha) = +\cosh\alpha$

$\cosh^2\theta - \sinh^2\theta = 1$

& $\cos^2\theta + \sin^2\theta = 1$

& $\cos^2\theta - \sin^2\theta = \cos 2\theta$

$\Rightarrow \cosh^2\theta + \sinh^2\theta = \cosh 2\theta$

So, $\cosh 2\theta = 2\cosh^2\theta - 1 = 2\sinh^2\theta + 1$

$$= \begin{vmatrix} \frac{\partial x}{\partial\alpha} & \frac{\partial x}{\partial\beta} \\ \frac{\partial y}{\partial\alpha} & \frac{\partial y}{\partial\beta} \end{vmatrix} = \begin{vmatrix} a\sinh\alpha\cos\beta & -a\cosh\alpha \\ a\cosh\alpha\sin\beta & a\sinh\alpha \end{vmatrix}$$

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$= a^2 \sinh^2\alpha \cos^2\beta + a^2 \cosh^2\alpha \sin^2\beta$

$= \frac{1}{4} a^2 \left[(2\sinh^2\alpha)(2\cos^2\beta) + (2\cosh^2\alpha)(2\sin^2\beta) \right]$

$= \frac{1}{4} a^2 \left[(\cosh 2\alpha - 1)(1+\cos 2\beta) + (\cosh 2\alpha + 1)(1-\cos 2\beta) \right]$

$= \frac{1}{4} a^2 \left[\cosh 2\alpha + \cosh 2\alpha \cos 2\beta - 1 - \cos 2\beta + \cosh 2\alpha - \cosh 2\alpha \cos 2\beta + 1 - \cos 2\beta \right]$

$= \frac{1}{4} a^2 \left[2\cosh 2\alpha - 2\cos 2\beta \right]$.

$$\frac{\partial(x,y)}{\partial(\alpha,\beta)} = \frac{1}{2} a^2 \left[\cosh 2\alpha - \cos 2\beta \right]$$

Hence proved

~~$= \frac{1}{2} a^2$~~

I

(5) Q. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$, evaluate $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$.

⇒ Sol^{n}:- We have.

$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$

$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$ $\frac{\partial y}{\partial r} = \sin\theta\sin\phi$.

$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$ $\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$

$\frac{\partial x}{\partial \phi} = r\sin\theta(-\sin\phi)$ $\frac{\partial y}{\partial \phi} = r\cos\theta\cdot\cos\phi$

& $z = r\cos\theta$ $\frac{\partial z}{\partial \theta} = -r\sin\theta$.

$\therefore \frac{\partial z}{\partial r} = \cos\theta$ $\frac{\partial z}{\partial \phi} = 0$.

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Now, $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$

$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$

$= \sin\theta\cos\phi\,[0 + r^2\sin^2\theta\cos\phi]$

$- r\cos\theta\cos\phi\,[0 - r\sin\theta\cos\phi\cos\theta]$

$+ r\sin\theta\sin\phi\,[-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi]$

$= r^2[\sin^3\theta\cos^2\phi + \sin\theta\cos^2\theta\cos^2\phi + \sin\theta\cos^2\theta\sin^2\phi]$

$= r^2[\sin^3\theta(\cos^2\phi + \sin^2\phi) + \sin\theta\cos^2\theta(\cos^2\phi + \sin^2\phi)]$

$= r^2[\sin^3\theta + \sin\theta\cos^2\theta]$

$= r^2[\sin\theta(\sin^2\theta + \cos^2\theta)]$

$= r^2\sin\theta$

(3) **Case of functions of functions :-**

If $z_1, z_2, z_3 \ldots, z_n$ are functions of $y_1, y_2, y_3 \ldots$ and $y_1, y_2, y_3 \ldots, y_n$ are functions of $x_1, x_2, x_3 \ldots$

$$\frac{\partial(z_1,z_2,z_3,\ldots,z_n)}{\partial(x_1,x_2,x_3,\ldots,x_n)} = \frac{\partial(z_1,z_2,\ldots,z_n)}{\partial(y_1,y_2,y_3,\ldots,y_n)} \cdot \frac{\partial(y_1,y_2,\ldots,y_n)}{\partial(x_1,x_2,\ldots,x_n)}$$

We know that,

$$\frac{\partial z_1}{\partial x_1} = \frac{\partial z_1}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_1} + \frac{\partial z_1}{\partial y_2}\frac{\partial y_2}{\partial x_1} + \cdots + \frac{\partial z_1}{\partial y_n}\cdot\frac{\partial y_n}{\partial x_1}$$

$$\frac{\partial z_1}{\partial x_1} = \sum \frac{\partial z_1}{\partial y_r}\cdot\frac{\partial y_r}{\partial x_1} \quad \text{---(i)}$$

Similarly, $\frac{\partial z_1}{\partial x_2} = \sum \frac{\partial z_1}{\partial y_r}\cdot\frac{\partial y_r}{\partial x_2}$, $\frac{\partial z_1}{\partial x_3} = \sum \frac{\partial z_1}{\partial y_r}\cdot\frac{\partial y_r}{\partial x_3}$

and $\frac{\partial z_1}{\partial x_n} = \sum \frac{\partial z_1}{\partial y_r}\cdot\frac{\partial y_r}{\partial x_n}$.

Now, $\frac{\partial(z_1,z_2,\ldots,z_n)}{\partial(y_1,y_2,\ldots,y_n)}\cdot\frac{\partial(y_1,y_2,\ldots,y_n)}{\partial(x_1,x_2,\ldots,x_n)}$

$$= \begin{vmatrix} \frac{\partial z_1}{\partial y_1} & \frac{\partial z_1}{\partial y_2} & \cdots & \frac{\partial z_1}{\partial y_n} \\ \frac{\partial z_2}{\partial y_1} & \frac{\partial z_2}{\partial y_2} & \cdots & \frac{\partial z_2}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial z_n}{\partial y_1} & \frac{\partial z_n}{\partial y_2} & \cdots & \frac{\partial z_n}{\partial y_n} \end{vmatrix} \times \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial z_1}{\partial y_1}\times\frac{\partial y_1}{\partial x_1} + \frac{\partial z_1}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_1} + \cdots + \frac{\partial z_1}{\partial y_n}\cdot\frac{\partial y_n}{\partial x_1}, & \frac{\partial z_1}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_2} + \frac{\partial z_1}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_2} + \cdots + \frac{\partial z_1}{\partial y_n}\cdot\frac{\partial y_n}{\partial x_1}, & \cdots \\ \frac{\partial z_2}{\partial y_1}\times\frac{\partial y_1}{\partial x_1} + \frac{\partial z_2}{\partial y_2}\times\frac{\partial y_2}{\partial x_1} + \cdots + \frac{\partial z_2}{\partial y_n}\times\frac{\partial y_n}{\partial x_1}, & \frac{\partial z_2}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_2} + \frac{\partial z_2}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_2} + \cdots + \frac{\partial z_2}{\partial y_n}\frac{\partial y_n}{\partial x_2}, & \cdots \\ \cdots & \cdots & \cdots \\ \frac{\partial z_n}{\partial y_1}\times\frac{\partial y_1}{\partial x_1} + \frac{\partial z_n}{\partial y_2}\frac{\partial y_2}{\partial x_1} + \cdots + \frac{\partial z_n}{\partial y_n}\times\frac{\partial y_n}{\partial x_1}, & \frac{\partial z_n}{\partial y_1}\times\frac{\partial y_1}{\partial x_2} + \frac{\partial z_n}{\partial y_2}\times\frac{\partial y_2}{\partial x_2} + \cdots + \frac{\partial z_n}{\partial y_n}\times\frac{\partial y_n}{\partial x_2}, & \cdots \end{vmatrix}$$

$$= \begin{bmatrix} \sum \frac{\delta z_1}{\delta y_r} \cdot \frac{\delta y_r}{\delta x_1} & \sum \frac{\delta z_1}{\delta y_r} \cdot \frac{\delta y_r}{\delta x_2} & \cdots & \sum \frac{\delta z_1}{\delta y_r} \times \frac{\delta y_r}{\delta x_n} \\ \sum \frac{\partial z_2}{\delta y_r} \cdot \frac{\delta y_r}{\delta x_1} & \sum \frac{\delta z_2}{\delta y_r} \cdot \frac{\delta y_r}{\partial x_2} & \cdots & \sum \frac{\delta z_2}{\delta y_r} \times \frac{\delta y_r}{\delta x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \sum \frac{\delta z_n}{\delta y_r} \times \frac{\delta y_r}{\delta x_1} & \sum \frac{\partial z_n}{\delta y_r} \cdot \frac{\delta y_r}{\delta x_2} & \cdots & \sum \frac{\partial z_n}{\delta y_r} \times \frac{\partial y_r}{\delta x_n} \end{bmatrix}$$

$$= \begin{vmatrix} \frac{\delta z_1}{\delta x_1} & \frac{\delta z_1}{\delta x_2} & \cdots & \frac{\delta z_1}{\delta x_n} \\ \frac{\delta z_2}{\delta x_1} & \frac{\delta z_2}{\delta x_2} & \cdots & \frac{\delta z_2}{\delta x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\delta z_n}{\delta x_1} & \frac{\delta z_n}{\delta x_2} & \cdots & \delta z \end{vmatrix} \quad \text{[using the result} = \frac{\delta(z_1, z_2, \cdots \delta z_n)}{\delta(x_1, x_2, \cdots x_n)}$$

(i) If $u_1 = x_2^2 + 2a_1 x_2 x_3 + x_3^2$.

(ii) $u_2 = x_1^2 + 2a_2 x_3 x_1 + x_3^2$

(iii) $u_3 = x_1^2 + 2a_3 x_1 x_2 + x_2^2$, evaluate $\frac{\partial(u_1, u_2, u_3}{\partial(x_1, x_2,}$

⇒ Sol^{n}:- $\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$ —

Now,

Since, $u_1 = x_2^2 + 2a_1 x_2 x_3 + x_3^2$.

$\therefore \frac{\partial u_1}{\partial x_1} = 0$

$\Rightarrow \frac{\partial u_1}{\partial x_2} = 2x_2 + 2a_1 x_3 = 2(x_2 + a_1 x_3)$

& $\frac{\partial u_1}{\partial x_3} = 2a_1 x_2 + 2x_3 = 2(a_1 x_2 + x_3)$

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$$\therefore \frac{\partial u_2}{\partial x_1} = 2x_1 + 2a_2x_3 = 2(x_1 + a_2x_3)$$

$$\frac{\partial u_2}{\partial x_2} = 0 \quad \& \quad \frac{\partial u_2}{\partial x_3} = 2a_3x_1 + 2x_3$$

$$= 2(a_2x_1 + x_3).$$

And $u_3 = x_1^2 + 2a_3x_1x_2 + x_2^2$

$$\therefore \frac{\partial u_3}{\partial x_1} = 2x_1 + 2a_3x_2 = 2(x_1 + a_3x_2)$$

$$\frac{\partial u_3}{\partial x_2} = 2a_3x_1 + 2x_2 = 2(a_3x_1 + x_2).$$

$$\& \quad \frac{\partial u_3}{\partial x_3} = 0$$

From equo (i),

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} 0 & 2(x_2 + a_1x_3) & 2(a_1x_2 + x_3) \\ 2(x_1 + a_2x_3) & 0 & 2(a_2x_1 + x_3) \\ 2(x_1 + a_3x_2) & 2(a_3x_1 + x_2) & 0 \end{vmatrix}$$

$$= 0(\quad) - 2(x_2 + a_1x_3)[0 - 4(a_1x_2 + x_3)(x_1 + a_3x_2)] + 2(x_3 + a_1x_2)\,2(x_1 + a_2x_3)\,2(x_2 + a_1a_3x_1)$$

$$= 8(x_2 + a_1x_3)(x_3 + a_1x_2)(x_1 + a_3x_2) + 8(x_3 + a_1x_2)(x_1 + a_2x_3)(x_2 + a_1a_3x_1)$$

$$= 8[(x_2 + a_1x_3)(x_3 + a_1x_2)(x_1 + a_3x_2) + (x_3 + a_1x_2)(x_1 + a_2x_3)(x_2 + a_1a_3x_1)]$$

(ii)

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Jacobian of Implicit function

If the functions $z_1, z_2, \cdots, z_n$ and $x_1, x_2, \cdots x_n$ be connected by the equations

$f_1(z_1, z_2, \cdots, z_n, x_1, x_2, \cdots, x_n) = 0$

$f_2(z_1, z_2, \cdots, z_n, x_1, x_2, \cdots, x_n) = 0$ ...

$f_n(z_1, z_2, \cdots, z_n, x_1, x_2, \cdots, x_n) = 0$

i.e., $z_1, z_2, \cdots, z_n$ are implicit functions of $x_1, x_2, \cdots, x_n$ then differentiating,

We have,

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_1}{\partial z_1}\cdot\frac{\partial z_1}{\partial x_1} + \frac{\partial f_1}{\partial z_2}\cdot\frac{\partial z_2}{\partial x_1} + \cdots + \frac{\partial f_1}{\partial z_n}\cdot\frac{\partial z_n}{\partial x_1} = 0.$$

$$\Rightarrow \frac{\partial f_1}{\partial x_2} + \frac{\partial f_1}{\partial z_1}\cdot\frac{\partial z_1}{\partial x_2} + \frac{\partial f_1}{\partial z_2}\cdot\frac{\partial z_2}{\partial x_2} + \cdots + \frac{\partial f_1}{\partial z_n}\cdot\frac{\partial z_n}{\partial x_2} = 0,$$

Similarly,

$$\frac{\partial f_2}{\partial x_1} + \frac{\partial f_2}{\partial z_1}\cdot\frac{\partial z_1}{\partial x_1} + \frac{\partial f_2}{\partial z_2}\cdot\frac{\partial z_2}{\partial x_1} + \cdots + \frac{\partial f_2}{\partial z_n}\cdot\frac{\partial z_n}{\partial x_1} = 0$$

and $\frac{\partial f_n}{\partial x_1} + \frac{\partial f_n}{\partial z_1}\cdot\frac{\partial z_1}{\partial x_1} + \frac{\partial f_n}{\partial z_2}\cdot\frac{\partial z_2}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial z_n}\cdot\frac{\partial z_n}{\partial x_1} = 0$ etc.

Now, $\frac{\partial(f_1, f_2, \cdots, f_n)}{\partial(z_1, z_2, \cdots, z_n)} \cdot \frac{\partial(z_1, z_2, \cdots, z_n)}{\partial(x_1, x_2, \cdots, x_n)}$

$$= \begin{vmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} & \cdots & \frac{\partial f_2}{\partial z_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial z_1} & \frac{\partial f_n}{\partial z_2} & \cdots & \frac{\partial f_n}{\partial z_n} \end{vmatrix} \times \begin{vmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \cdots & \frac{\partial z_1}{\partial x_n} \\ \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial x_2} & \cdots & \frac{\partial z_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial z_n}{\partial x_1} & \frac{\partial z_n}{\partial x_2} & \cdots & \frac{\partial z_n}{\partial x_n} \end{vmatrix}$$

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After multiplication,

$$= \begin{vmatrix} \sum \frac{\partial f_1}{\partial z_r} \cdot \frac{\partial z_r}{\partial x_1} & \sum \frac{\partial f_1}{\partial z_r} \cdot \frac{\partial z_r}{\partial x_2} & \cdots & \sum \frac{\partial f_1}{\partial z_r} \cdot \frac{\partial z_r}{\partial z_n} \\ \sum \frac{\partial f_2}{\partial z_r} \cdot \frac{\partial z_r}{\partial x_1} & \sum \frac{\partial f_2}{\partial z_r} \cdot \frac{\partial z_r}{\partial x_2} & \cdots & \sum \frac{\partial f_2}{\partial z_r} \cdot \frac{\partial z_r}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \sum \frac{\partial f_n}{\partial z_r} \cdot \frac{\partial z_r}{\partial x_1} & \sum \frac{\partial f_n}{\partial z_r} \cdot \frac{\partial z_n}{\partial x_2} & \cdots & \sum \frac{\partial f_n}{\partial z_r} \cdot \frac{\partial z_r}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{\partial f_1}{\partial x_1} & -\frac{\partial f_1}{\partial x_2} & \cdots & -\frac{\partial f_1}{\partial x_n} \\ -\frac{\partial f_2}{\partial x_1} & -\frac{\partial f_2}{\partial x_2} & \cdots & -\frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{\partial f_n}{\partial x_1} & -\frac{\partial f_n}{\partial x_2} & \cdots & -\frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

Since, $\frac{\partial f_1}{\partial x_1} + \sum \frac{\partial f_1}{\partial z_r} \cdot \frac{\partial z_r}{\partial x_1} = 0$

$$\therefore \sum \frac{\partial f_1}{\partial z_r} \cdot \frac{\partial z_r}{\partial x_1} = -\frac{\partial f_1}{\partial x_1}$$ (proved above)

$$\therefore \frac{\partial(f_1, f_2, \cdots, f_n)}{\partial(z_1, z_2, \cdots, z_n)} \cdot \frac{\partial(z_1, z_2, \cdots, z_n)}{\partial(x_1, x_2, \cdots, x_n)} = (-1)^n \frac{\partial(f_1, f_2, \cdots, f_n)}{\partial(x_1, x_2, \cdots, x_n)}$$

Hence, $\frac{\partial(z_1, z_2, \cdots, z_n)}{\partial(x_1, x_2, \cdots, x_n)} =$

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If $J = \frac{\partial(u,v)}{\partial(x,y)}$ & $J' = \frac{\partial(x,y)}{\partial(u,v)}$, prove that $J \cdot J' = 1$.

Soln:- $\because J \cdot J' = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)}$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial x}\cdot\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} \end{vmatrix} \quad \text{--- (i)}$$

Now, let $u = f_1(x,y)$ --- (ii) & $v = f_2(x,y)$.
$v = v(x,y)$ --- (iii)

Then, diff (ii) w.r.t. u & v, we get

$$\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} = 0$$

So, $\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} = 0$

Similarly, from eqn (iii), we get

$$\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} = 0$$

& $\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} = 1$.

Substituting these values in (i), we get

$$\boxed{J \cdot J' = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1}$$

Hence, proved

Infinite Series

Real Analysis / Numbers

- Rational
  - (-ve)
  - zero
  - (+ve)
- Irrational

Series :- It is the sum of the terms of a sequence is called series.

There are 2 types of series :—

(i) **Infinite :-** A series having infinite no. of terms is called infinte series.

(ii) **Finite :-** A series having finite no. of terms is called finite series.

⇒ **Sequence :-** If each +ve integers (1, 2, 3, ---) there corresponds a finite real nos $a_1, a_2, a_3, \dots$ are said to be a sequence.

A sequence is said to be finite or infinite according as the no. of elements finite or infinite.

A sequence of elements $a_1, a_2, a_3, \dots, a_n$ is denoted by $\langle a_n \rangle$, (a_n) or $\{a_n\}$.

Convergency & Divergency

Convergency :— If $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \dots + u_n + \dots$ is an infinite series, then if

(a) $\lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{n=1}^{\infty} u_n = k$ (a definite no.)

then the series $\sum_{n=1}^{\infty} u_n$ is said to be convergent.

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(b) **Divergency**:- If $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{n=1}^{\infty} u_n = \pm\infty$, then the series $\sum_{n=1}^{\infty} u_n$ is said to be divergent.

(c) If $\lim_{n\to\infty} S_n = \pm l$, i.e., $\lim_{n\to\infty} S_n = l$ or $\lim_{n\to\infty} S_n = -l$ then, the series or $\sum_{n=1}^{\infty} u_n$ is called an oscillatory

Theorem:- An Alternating Series $\sum_{1}^{\infty} (-1)^{n-1} u_n$ is convergent if (i) $u_n > u_{n+1}$ or $u_{n+1} < u_n$ & (ii) $\lim_{n\to\infty} u_n = 0$

Proof:-

$\because \sum_{1}^{\infty} (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \cdots \infty$

where, $u_1 > u_2 > u_3 > u_4 > \cdots$.

Let $S_n = u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \cdots$ up to n terms

$\therefore S_{2m} = (u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \cdots + (u_{2m-1} - u_{2m})$

$\Rightarrow S_{2m} = (+ve) > 0$

$S_{2m} \geq 0$ — (i)

Since, $S_{2m} = u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \cdots + \cdots - u_{2m}$

$\Rightarrow S_{2m} = u_1 - [(u_2 - u_3) + (u_4 - u_5) + (u_6 - u_7) + \cdots + \cdots]$

$\Rightarrow S_{2m} = u_1 - (+ve \text{ numbers})$

$\therefore S_{2m} < u_1$ — (ii)

i.e., $S_{2m} = L$ (say) $< u_1$ — (iii)

From equ (i) & (iii) $0 < S_{2m} = L < u_1$

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Taking lim on both sides, we get

$$\lim_{m\to\infty} S_{2m} = L \quad \text{---} (iv)$$

Again, $S_{2m+1} = u_1 - u_2 + u_3 - u_4 + \cdots - u_{2m}$.

∴ From equ (a),

$$\Rightarrow \boxed{S_{2m+1} = S_{2m} + u_{2m+1}}$$

Taking lim on both sides, we get

$$\lim_{m\to\infty} S_{2m+1} = \lim_{m\to\infty} S_{2m} + \lim_{m\to\infty} u_{2m+1}$$

$$\boxed{\lim_{m\to\infty} S_{2m+1} = L + 0}$$ [from condition (ii) as $\lim_{n\to\infty} u_n = 0$]

$$\Rightarrow \boxed{S_{2m+1} = L} \quad \text{---} (v)$$

Hence, from equ (iv) & (v), we get

$$\boxed{\lim_{n\to\infty} S_n = L}$$ (a definite no.)

Therefore, $\boxed{\sum_1^\infty (-1)^{n-1} u_n}$ is convergent

proved

State & Prove Leibnitz's test

Statement:- Prove that the alternating series

$\boxed{\sum_1^\infty (-1)^{n-1} u_n}$ is convergent if

(i) Each term is numerically less than preceding, i.e., $\boxed{|u_{n+1}| < |u_n| \ \forall\ n.}$

(ii) $\boxed{u_n \to 0 \text{ as } n \to \infty}$

OR

An infinite series $\sum_1^\infty u_n$ is convergent if

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State and prove Comparison test :-

22/11/18

Statement :- If $\sum u_n$ & $\sum v_n$ be two infinite series in which the ratio of the corresponding terms are all finite, then the 2 series are both convergent or divergent.

Proof :- Let $\sum u_n = u_1 + u_2 + u_3 + u_4 + u_5 + \cdots u_n$

& $\sum v_n = v_1 + v_2 + v_3 + \cdots + v_n + \cdots \infty$

be two infinite series in which $\frac{u_r}{v_r}$ are all finite, where $(r = 1, 2, 3, \cdots)$.

Let S_n & T_n denotes the sum of n^{th} terms the series u_n & v_n, then.

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n$$

& $$T_n = v_1 + v_2 + v_3 + \cdots + v_n.$$

Series u_r/v_r is always finite, the first n

$$\frac{u_1}{v_1}, \frac{u_2}{v_2}, \frac{u_3}{v_3}, \cdots, \frac{u_n}{v_n}.$$

Let λ & μ be the greatest & least of the ratio, then

$$\lambda \geq \frac{u_1}{v_1} \geq \mu \Rightarrow \lambda v_1 \geq u_1 \geq \mu v_1$$

$$\lambda \geq \frac{u_2}{v_2} \geq \mu \Rightarrow \lambda v_2 \geq u_2 \geq \mu v_2$$

$$\lambda \geq \frac{u_3}{v_3} \geq \mu \Rightarrow \lambda v_3 \geq u_3 \geq \mu v_3.$$

$$\lambda \geq \frac{u_n}{v_n} \geq \mu \Rightarrow \lambda v_n \geq u_n \geq \mu v_n.$$

Adding vertically, we get.

$$\Rightarrow \lambda = (v_1 + v_2 + v_3 + \cdots + v_n) \geq (u_1 + u_2 + u_3 + \cdots + u_n) \geq \mu(v_1 + v_2 + \cdots$$

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or $\lambda \geq \frac{u_1+u_2+u_3+\cdots+u_n}{v_1+v_2+v_3+\cdots+v_n} \geq \mu$.

or $\boxed{\lambda \geq \frac{S_n}{T_n} \geq \mu}$.

Thus $\frac{S_n}{T_n}$ is finite as λ & μ are finite.

Let $\boxed{S_n/T_n = k}$, where k is finite. or.

$$\boxed{S_n = kT_n \quad \text{or} \quad \lim_{n\to\infty} S_n = k \cdot \lim_{n\to\infty} T_n} \quad \text{— (i)}$$

Now, we suppose that $\sum v_n$ is convergent, then

$\lim_{n\to\infty} T_n$ will be a finite.

And as k is a finite.

From equo (i), $\boxed{\lim_{n\to\infty} S_n \text{ is also finite.}}$

And hence, $\boxed{\sum u_n \text{ is convergent}}$ (i) proved

If $\sum v_n$ is divergent, then,

$\lim_{n\to\infty} T_n$ is infinite [i.e., $\lim_{n\to\infty} T_n = \infty$]

So, from equo (i), $\boxed{\lim_{n\to\infty} S_n = \infty}$

And hence, $\boxed{\sum u_n \text{ is also divergent.}}$ proved

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2nd form of Comparison test :-

If $\sum u_n$ & $\sum v_n$ be two infinite series, t

$\sum u_n$ will be convergent, if

$$\boxed{\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}}}$$ and $\sum v_n$ is con

Also, $\sum u_n$ will be divergent if.

$$\boxed{\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}}$$

P-series or auxiliary Series :-

Statement :- Prove that the series

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots}$$

is (i) convergent if $p > 1$.

(ii) divergent if $p \leq 1$.

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Geometric Series :-

G.P is $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots \infty$.

$= \frac{a}{1-r}$ is (i) Convergent if $r < 1$.

(ii) divergent if $r \geq 1$.

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★ D'Alembert's Ratio Test :-

Statement :- An infinite series $\sum U_n$ of +ve terms is convergent if from and after a particular term (fixed term), the ratio of each term to the preceeding is always less than unity.

The series is divergent if this ratio is greater than or equal to unity.

OR.

P.T. the series $\sum U_n$ of positive terms is convergent if $\frac{U_n}{U_{n-1}} < k < 1$.

where k is a fixed no. and 'n' has any value, ~~then~~ the series is divergent if

$$\frac{U_n}{U_{n-1}} \geq 1.$$

Proof :- Let the series from the particular term be $U_1 + U_2 + U_3 + \cdots + U_n + \cdots \infty$, in which $\frac{U_s}{U_{s-1}} < k$ (say), where $(k < 1)$.

Let S ~~be~~ denotes the sum of the series

then $S = U_1 + U_2 + U_3 + \cdots + \infty$

$\leq u_1\left[1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\cdots\right]$.

$= u_1\left[1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\cdots\right]$.

but from the given condition:

$$\boxed{\frac{u_2}{u_1}<1,\ \frac{u_3}{u_2}<1,\ \frac{u_4}{u_3}<1}$$

Therefore $S < u_1\left[1+k+k\cdot k+k\cdot k\cdot k+\cdots\right]$

$\Rightarrow S < u_1\left[1+k+\underset{<k}{k^2}+\underset{<1}{k^3}+\cdots\right]$

By using G.P

$$\boxed{S < u_1\cdot\frac{1}{1-k}} \Rightarrow \boxed{S < \frac{u_1}{1-k}}$$ → which is finite as $\boxed{k<1}$

Thus, the sum of the series is, being less than a finite series, is also finite

And hence, the series is convergent

Case II:-

let $\frac{u_n}{u_{n-1}} \geq 1$, i.e., $\frac{u_2}{u_1}\geq 1, \frac{u_3}{u_2}\geq 1, \frac{u_4}{u_3}\geq 1$...

i.e., $u_2 \geq u_1, u_3 \geq u_2, u_4 \geq u_3, \cdots$

So, $u_1 \leq u_2, u_2 \leq u_3 \leq u_3 \leq u_4 \leq \cdots$

i.e., $\boxed{u_1 \leq u_2 \leq u_3 \leq u_4 \leq u_5 \leq \cdots}$

$\therefore \boxed{S = u_1+u_2+u_3+u_4+\cdots \infty \geq u_1+u_1+u_1+u_1+\cdots}$

Also, $S_n = u_1+u_2+u_3+\cdots+u_n \geq u_1+u_1+u_1+\cdots+u_1$ up (to n terms).

$\therefore \boxed{S_n \geq nu}$.

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JMS

Logarithmic Test

State and prove logarithmic test

Statement :-

If $\sum u_n$ be a series of +ve terms, then $\sum u_n$ will be convergent or divergent if according as

$$\lim_{n\to\infty} n \log \left|\frac{u_n}{u_{n+1}}\right| > \text{or} < 1$$

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⇒ **Proof :-** Let us compare the given series $\sum u_n$ with the auxiliary series $\sum v_n$; where

$$\sum v_n = \sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots\cdots$$

We know that $\sum v_n$ is convergent or divergent according as $(p > 1)$ or $(p \leq 1)$.

Case I :- By Comparison Test, the given series $\sum u_n$ will be convergent, if

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} \Rightarrow \frac{u_n}{u_{n+1}} > \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}} \quad \left[\because v_n = \frac{1}{n^p} \text{ & } v_{n+1} = \frac{1}{(n+1)^p}\right]$$

$$> \frac{(n+1)^p}{n^p} \Rightarrow \left(\frac{n+1}{n}\right)^p$$

$$\Rightarrow \frac{u_n}{u_{n+1}} > \left(1 + \frac{1}{n}\right)^p$$

Taking log on both sides, we get

$$\log \frac{u_n}{u_{n+1}} > \log \left(1 + \frac{1}{n}\right)^p$$

$$\text{" } > p \log \left(1 + \frac{1}{n}\right)$$

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$\because \boxed{\log(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots}$

$\therefore \log \frac{u_n}{u_{n+1}} > p\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \cdots\right]$

$\Rightarrow \log \frac{u_n}{u_{n+1}} > \frac{p}{n}\left[1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + \cdots\right]$

$\Rightarrow n \log \frac{u_n}{u_{n+1}} > p\left[1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + \cdots\right]$

Taking limits on both sides, when $\boxed{n \to \infty}$

$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} > p \lim_{n\to\infty}\left[1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + \cdots\right]$

$> p[1 - 0 + 0 - \cdots]$

$> p > 1 \quad [\text{as } p > 1]$

Therefore,

$$\boxed{\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} > 1}$$

proved

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8/12/23

Infinite Series

Raabe's Test

State and prove Raabe's test.

⇒ Statement:-

If $\sum u_n$ be a series of +ve term then $\sum u_n$ will be convergent or divergent according as

$$\boxed{\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}-1\right) > 1 \text{ or } < 1}$$

⇒ Proof:- We combine the given series with an imaginary series.

$\boxed{\sum v_n = \sum \frac{1}{n^p}}$ which is convergent if $(p>1)$ and divergent if $(p\le 1)$.

P-series

$$\sum v_n = \sum \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

(i) $p > 1$
(ii) $p \le 1$

Case I:- Suppose $\sum v_n$ is convergent then,

$$\boxed{\sum v_n = \sum \frac{1}{n^p}} \; ; \; p > 1$$

Again from the **comparison test**,

$\sum u_n$ will be convergent if

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} \Rightarrow \boxed{\frac{u_n}{u_{n+1}} > \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}}}$$

$$\left[\because v_n = \frac{1}{n^p} \;\&\; v_{n+1} = \frac{1}{(n+1)^p}\right]$$

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$\Rightarrow \boxed{\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p}}$ *, so, $\boxed{\frac{u_n}{u_{n+1}} > \left(\frac{n+1}{n}\right)^p}$

$\Rightarrow \boxed{\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)^p}$

So, $\frac{u_n}{u_{n+1}} > 1 + p\frac{1}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2} + \frac{p(p-1)(p-2)}{3!}\frac{1}{n^3} + \cdots$

[by expanding by Binomial theorem]

$\Rightarrow \left(\frac{u_n}{u_{n+1}} - 1\right) > p\frac{1}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2} + \frac{p(p-1)(p-2)}{3!}\frac{1}{n^3} + \cdots$

$\Rightarrow \left(\frac{u_n}{u_{n+1}} - 1\right) > \frac{1}{n}\left[p + \frac{p(p-1)}{2!}\frac{1}{n} + \frac{p(p-1)(p-2)}{3!}\frac{1}{n^2} + \cdots\right]$

$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) > \left[p + \frac{p(p-1)}{2!}\cdot\frac{1}{n} + \frac{p(p-1)(p-2)}{3!}\cdot\frac{1}{n^2} + \cdots\right.$

Taking limits both sides $\boxed{n \to \infty}$, ~~#~~

So, $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > \lim_{n\to\infty}\left[p + \frac{p(p-1)}{2!}\frac{1}{n} + \cdots\right]$

$\Rightarrow \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + \frac{p(p-1)}{2!}\frac{1}{\infty} + \frac{p(p-1)(p-2)}{3!}\frac{1}{\infty^2} + \cdots\Big]$

$\Rightarrow \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > p > 1$ [as $p > 1$]

$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1$

Thus, Σu_n will be convergent if

$$\boxed{\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1}$$

Hence, proved

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Case II :- We suppose that $\boxed{\sum v_n = \sum \frac{1}{n^p}}$ is divergent is less than 0 $(p < 1)$.

Now, comparison test $\sum v_n$ will be divergent if $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$

$$\Rightarrow \frac{u_n}{u_{n+1}} < \frac{1/n^p}{1/(n+1)^p} \Rightarrow \frac{u_n}{u_{n+1}} < \frac{(n+1)^p}{n^p}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} < \left(\frac{n+1}{n}\right)^p \Rightarrow \frac{u_n}{u_{n+1}} < \left(1+\frac{1}{n}\right)^p$$

So, expanding by Binomial theorem, we get

$$\frac{u_n}{u_{n+1}} < 1 + p\cdot\frac{1}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2} + \text{- - - - -}$$

$$\Rightarrow \left(\frac{u_n}{u_{n+1}} - 1\right) < \left[p\cdot\frac{1}{n} + \frac{p(p-1)}{2!}\cdot\frac{1}{n^2} + \text{- - - - -}\right]$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) < \left[p + \frac{p(p-1)}{2!}\cdot\frac{1}{n} + \frac{p(p-1)(p-2)}{3!}\cdot\frac{1}{n^2} + \text{- -}\right]$$

Taking limits on both sides, we get $(n \to \infty)$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) < \lim_{n\to\infty}\left[p + \frac{p(p-1)}{2!}\cdot\frac{1}{n} + \text{- - - -}\right]$$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) < p + \frac{p(p-1)}{2!}\cdot\frac{1}{\infty} + \text{- - -}$$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) < p \leq 1 \text{ , as } (p \leq 1).$$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) \leq 1$$

Thus, $\sum v_n$ will be divergent if

$$\boxed{\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) \leq 1}$$

Hence, proved.

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09/01/24

Cauchy's Root Test

State and prove Cauchy root Test

⇒ **Statement:-** A series $\sum u_n$ of the +ve terms is convergent if $\boxed{\sqrt[n]{u_n} = u_n^{\frac{1}{n}} < k < 1}$ and the series is divergent if $\boxed{u_n^{\frac{1}{n}} \geq 1}$

⇒ **Proof:-**

Let $\boxed{u_n^{\frac{1}{n}} < k}$, i.e., $u_n < k^n$

$\therefore u_1 < k^1$

$u_2 < k^2$

$u_3 < k^3$

$u_4 < k^4$

So, $[u_1 + u_2 + u_3 + u_4 + \cdots] < [k + k^2 + k^3 + k^4 + \cdots]$

$< k[1 + k + k^2 + k^3 + \cdots]$

$< k\left(\frac{1}{1-k}\right)$ [by G.P.]

$< \frac{k}{1-k}$ which is finite

as $\boxed{k < 1}$

$\frac{1}{2} < 1$

$\Rightarrow \frac{k}{1-k} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1.$

Hence, the series is convergent proved

V.V.I

Case II :- Let $\boxed{u_n^{\frac{1}{n}} \geq 1}$.

i.e., $u_n \geq 1^n$.

$\therefore u_1 \geq 1, u_2 \geq 1, u_3 \geq 1, u_4 \geq 1, \ldots, u_n \geq 1$.

$\therefore u_1 + u_2 + u_3 + u_4 + \ldots + u_n \geq 1+1+1+\ldots$ up to n terms.

$\Rightarrow \boxed{S_n \geq n.1}$

Taking limits on both sides, as $n \to \infty$, we get

$$\lim_{n\to\infty} S_n \geq \lim_{n\to\infty} n.1 = \infty.1$$

$$\geq \infty.$$

Hence, the series is divergent. proved

Test for the convergent of the series, whose general term is $\frac{n^{n^2}}{(n+1)^{n^2}}$.

Soln:- Let the general term of the series is denoted by u_n.

So, $u_n = \frac{n^{n^2}}{(n+1)^{n^2}} = \left(\frac{n}{n+1}\right)^{n^2} = \left(\frac{1}{\frac{(n+1)}{n}}\right)^{n^2}$

$\Rightarrow u_n = \frac{1^{n^2}}{\left(\frac{n+1}{n}\right)^{n^2}}$

$\therefore u_n = \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$

Taking nth root both sides, we get

i.e., $u_n^{\frac{1}{n}} = \frac{1}{\left(1+\frac{1}{n}\right)^{n\times\frac{1}{n}}} = \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$ (i).

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Taking limit both sides when $n \to \infty$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \qquad \left[\because \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e\right].$$

$$\Rightarrow \lim_{n\to\infty} u_n^{\frac{1}{n}} = \frac{1}{e} < 1. \quad (2 < e < 3 \to 2.7).$$

i.e., $u_n^{\frac{1}{n}} < 1.$

Hence, from Cauchy root Test, the series will be convergent. Ans

V.V.I

Q-2. Find whether the series $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots$ is convergent or divergent.

⇒ Soln:- The general term of the given series is

$$u_n = \left[\frac{(n+1)^{n+1}}{n^{(n+1)}} - \frac{(n+1)}{n}\right]^{-n}$$

V.V.I

where, $n = 1, 2, 3, 4, \cdots$ & $(n \to \infty)$

Taking n^{th} root both sides, we get

$$u_n^{\frac{1}{n}} = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{(n+1)}{n}\right]^{-1} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-1}$$

$$\Rightarrow u_n^{\frac{1}{n}} = \left[\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$\Rightarrow u_n^{\frac{1}{n}} = \left[\left(1+\frac{1}{n}\right)^{1} \cdot \left(1+\frac{1}{n}\right)^{n} - \left(1+\frac{1}{n}\right)\right]^{-1}$$

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Taking limits both sides as $n \to \infty$.

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right)^n - \left(1+\frac{1}{n}\right)\right]^{-1}.$$

$$= \left[e(1+0)(1+0)^n - (1+0)\right]^{-1}.$$

$$= \left[(1+0)e - (1+0)\right]^{-1}$$

$$= \boxed{(e-1)^{-1} = \frac{1}{e-1} < 1} \cdot [e = 2.7]$$

$$\Rightarrow \boxed{\lim_{n\to\infty} u_n^{\frac{1}{n}} = \frac{1}{e-1} < 1}$$

Hence, the series will be convergent by Cauchy Root Test.

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11/01/24

Absolutely Convergent & Conditionally Convergent or Semi-Convergent

Statement:-

The series $\sum u_n$ containing +ve & -ve terms is said to be absolutely convergent, if the series $\sum |u_n|$ is convergent

(or)

An infinite series $\sum u_n$ is said to be absolutely convergent if $\sum |u_n|$ is convergent

(Or)

The series which remains convergent when all the (-ve) terms in it are made (+ve), then $\sum u_n$ is called absolutely convergent

(OR)

If $\sum u_n$ is convergent and $\sum |u_n|$ is divergent then $\sum u_n$ is said to be conditionally convergent or semi-convergent.

Eg:- ① [any alternating series is convergent]

(1) ⇒ **Proof:-** Consider a series:

$$\sum u_n = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \infty \quad \text{— (i)}$$

The above series is convergent being an alternating series.

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The series after making all (-ve) terms (+ve) then

$$\sum |u_n| = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \infty$$

$$= \sum \frac{1}{n^2}$$ [Compare it with $\sum \frac{1}{n^p}$. Hence $p = 2 > 1$]

which is again convergent.

Hence, the given series is absolutely convergent proved

Ex:-②

Again Consider the series.

$$\sum u_n = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \infty \quad — ②$$

which is convergent being an alternating series

The series after making all (-ve) terms (+ve),

then $\sum |u_n| = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \infty$

$$= \sum \frac{1}{n}$$ [Compare it with $\sum \frac{1}{n^p}$ is divergent if $p \leq 1$]

Then, the series be divergent as $p = 1$.

Hence, the series ② is conditionally convergent or semiconvergent. proved

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Theorem:- An absolutely convergent series ~~is said~~ itself convergent but the converse is not necessarily true.

Proof:- Let $\sum u_n$ be an absolutely convergent. Then by definition of absolute convergent $\sum |u_n|$ is also convergent.

Now $\boxed{\begin{matrix} u_n + |u_n| = 2|u_n| \\ = 0 \end{matrix}}$ $\begin{matrix} \} \text{if } u_n \text{ is positive,} \\ \text{if } u_n \text{ is (-ve).} \end{matrix}$

$\therefore u_n + |u_n| \geq 0 \;, \forall\, n$

Now we consider the series $\sum (u_n + |u_n|)$, then clearly it is a series with all (+ve) terms with less than or equal to the series $2\sum |u_n|$.

i.e., ~~$2\sum |u_n| \leq$~~ $\boxed{\sum (u_n + |u_n|) \leq 2\sum |u_n|}$

Therefore, $(u_n + |u_n|)$ is convergent as $\sum 2|u_n|$ is convergent.

Hence, $\sum (u_n + |u_n|)$ is convergent. Consequently $\sum u_n$ is convergent.

Converse:-

Again for the converse of the above theorem is not necessarily true.

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For example :- Consider an series

$\sum u_n = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \infty$

which is convergent being an absolutely convergent alternating series.

The series after making all (-ve) terms (+ve)

then $\sum |u_n| = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \infty$

$= \sum \frac{1}{n}$ [Compare it with $\sum \frac{1}{n^p}$ is divergent if $p \leq 1$]

Then, the series be divergent as $\boxed{p = 1}$

Hence, the series is conditionally convergent or semiconvergent. proved

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Pg 3

Test the series of convergence

$$1+\frac{2^2}{3^2}x^2+\frac{2^2\cdot4^2}{3^2\cdot5^2}x^2+\frac{2^2\cdot4^2\cdot6^2}{3\cdot5^2\cdot7^2}x^3+\cdots\cdots\infty.$$

⇒ Soln:-

Note:- (De Morgan's and Bertrand's test)

$$\left[n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right]\log \begin{cases}>1\\<1.\end{cases}$$

(i) $\frac{u_n}{u_{n+1}}<k<1$; convergent — D'Alembert's Test

(ii) $\frac{u_n}{u_{n+1}}\geq 1$; divergent

⇒ Soln:- Let the general or n^{th} terms of the given series be denoted by u_n. Then

$$u_n=\frac{2^2\cdot4^2\cdot6^2\cdots(2n)^2}{3^2\cdot5^2\cdot7^2\cdots(2n+1)^2}\cdot x^n.$$

So, $$u_{n+1}=\frac{2^2\cdot4^2\cdot6^2\cdots(2n)^2\{2(n+1)\}^2}{3^2\cdot5^2\cdot7^2\cdots(2n+1)^2\{2(n+1)+1\}^2}\cdot x^{n+1}$$

$$\therefore \frac{u_n}{u_{n+1}}=\frac{2^2\cdot4^2\cdot6^2\cdots(2n)^2}{3^2\cdot5^2\cdot7^2\cdots(2n+1)^2}\cdot\frac{x^n\times 3^2\cdot5^2\cdots(2n+1)^2\{2(n+1)+1\}^2}{\times 2^2\cdot4^2\cdot6^2\cdots(2n)^2\{2(n+1)\}^2}$$

$$\Rightarrow \frac{u_n}{u_{n+1}}=\frac{\{(2n+3)\}^2}{(2n+2)^2}\cdot\frac{1}{x}.=\frac{n^2\left(2+\frac{3}{n}\right)^2}{4n^2\left(1+\frac{1}{n}\right)^2}\cdot\frac{1}{x}.$$

Taking limit on both sides, we get

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{(2n+3)^2}{(2n+2)^2}\cdot\frac{1}{x}.=\frac{4}{4}\cdot\frac{1}{x}.$$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{\left(2+\frac{3}{n}\right)^2}{\left(2+\frac{2}{n}\right)^2}\cdot\frac{1}{x}.=\frac{4}{4}\cdot\frac{1}{x}=\frac{1}{x}.$$

Hence, by D'Alembert's ratio test, the given series is convergent or divergent according as

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$\frac{1}{x} > 1$ or $\frac{1}{x} < 1$.

i.e., $1 > x$ or $1 < x$.

$\Rightarrow$ $x < 1$ or $x > 1$.

$\therefore$ By D'Alembert's

$\frac{u_n}{u_{n-1}} < k < 1$.

$\therefore \frac{u_{n-1}}{u_n} > k > 1$.

When $\boxed{x = 1}$, then D'Alembert's test fails.

Now, Raabe's test will be helpful.

Since, $\frac{u_n}{u_{n+1}} = \frac{(2n+3)^2}{(2n+2)^2} \cdot \frac{1}{x}$.

$\because x = 1$

$\therefore \boxed{\frac{u_n}{u_{n+1}} = \frac{(2n+3)^2}{(2n+2)^2}}$

$\Rightarrow \frac{u_n}{u_{n+1}} - 1 = \frac{(2n+3)^2}{(2n+2)^2} - 1 = \frac{(2n+3)^2 - (2n+2)^2}{(2n+2)^2}$.

$\Rightarrow \frac{u_n}{u_{n+1}} - 1 = \frac{4n^2 + 9 + 12n - 4n^2 - 8n - 4}{(2n+2)^2} = \frac{4n+5}{(2n+2)^2}$

$\Rightarrow \frac{u_n}{u_{n+1}} - 1 = \frac{n\left(4 + \frac{5}{n}\right)}{n^2\left(2 + \frac{2}{n}\right)^2}$

$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{4 + \frac{5}{n}}{\left(2 + \frac{2}{n}\right)^2}$

Taking limit as $n \to \infty$, we get

$\Rightarrow \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{4 + \frac{5}{n}}{\left(2 + \frac{2}{n}\right)^2} = \frac{4}{4} = 1$

Hence, Raabe's test is also helpless to give any conclusion.

i.e., we can say test fails.

Therefore, $\because \frac{u_n}{u_{n+1}} - 1 = \frac{4n+5}{(2n+2)^2}$.

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By multiplying n both sides, we get

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = \frac{4n^2+5n}{(2n+2)^2}$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}}-1\right)-1 = \frac{4n^2+5n-4n^2-4-8n}{(2n+2)^2}$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}}-1\right)-1 = \frac{-4-3n}{(2n+2)^2} = \frac{-n(3+\frac{4}{n})}{n^2(2+\frac{2}{n})^2}$$

$$= \frac{-(3+\frac{4}{n})}{n(2+\frac{2}{n})^2}$$

Multiplying by log on both sides, we get

$$\log\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n = -\log\frac{(3+\frac{4}{n})}{n(2+\frac{2}{n})^2}\cdot\log n.$$

$$= \frac{-1}{n}\frac{(3+\frac{4}{n})}{(2+\frac{2}{n})^2}\cdot\frac{\log n}{n}$$

Taking limit both sides as $n\to\infty$, we get

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n = -\lim_{n\to\infty}\frac{\lim_{n\to\infty}(3+\frac{4}{n})}{\lim_{n\to\infty}(2+\frac{2}{n})^2}\cdot\frac{\log n}{n}$$

$$= \frac{3}{2}\times 0 = 0 < 1.$$

i.e., $\boxed{\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)^{\alpha}-1\right\}\log n < 1}$ $\left[\because \frac{\log n}{n}=0\right]$

Hence, by De' Morgan's & Bertrand's test, the given series is divergent when $x=1$.

∴

Determine the convergence of the series:

$$2x + \frac{3x^2}{8} + \frac{4x^3}{27} + \cdots + \frac{(n+1)}{n^3}x^n + \cdots \infty.$$

Soln:- Let the n^{th} term of the given series be denoted by u_n. Then

$$\boxed{u_n = \frac{(n+1)}{n^3}x^n} \quad \text{— (i)}$$

So, $u_{n+1} = \frac{(n+2)}{(n+1)^3}x^{n+1}$. — (ii)

$\Rightarrow$ (i) $\div$ (ii).

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)\,x^n\,(n+1)^3}{n^3\,(n+2)\,x^n \cdot x} = \frac{(n+1)^4}{n^3(n+2)} \cdot \frac{1}{x}.$$

$$= \frac{n^4\left(1+\frac{1}{n}\right)^4}{n^4\left(1+\frac{2}{n}\right)} \cdot \frac{1}{x}.$$

Taking limit both sides, we get (as $n \to \infty$)

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^4}{\left(1+\frac{2}{n}\right)} \cdot \frac{1}{x} = \frac{1}{x}.$$

Hence, by D'Alembert's ratio test, the given series is ~~divergent~~ convergent or divergent as $\frac{1}{x} > 1$ or $\frac{1}{x} < 1$.

$\Rightarrow x < 1$ or $x > 1$.

When $\boxed{x = 1}$, D'Alembert's test fails.

So, now we apply Raabe's test, we get

$\because \frac{u_n}{u_{n+1}} = \frac{(n+1)^4}{n^3(n+2)} \cdot \frac{1}{x}$. Also, since, $(x=1)$, so,

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$$\therefore \boxed{\frac{u_n}{u_{n+1}} = \frac{(n+1)^4}{n^3(n+2)}}.$$

$$\Rightarrow \frac{u_n}{u_{n+1}} - 1 = \frac{(n+1)^4 - n^3(n+2)}{n^3(n+2)} = \frac{n^4+1+2n-n^4-2n^3}{n^3(n+2)}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} - 1 = \frac{-2n^3+2n+1}{n^3(n+2)} = \frac{n^3\left(-2+\frac{2}{n^2}+\frac{1}{n^3}\right)}{n^3(n+2)}$$

Taking limit both sides, we get

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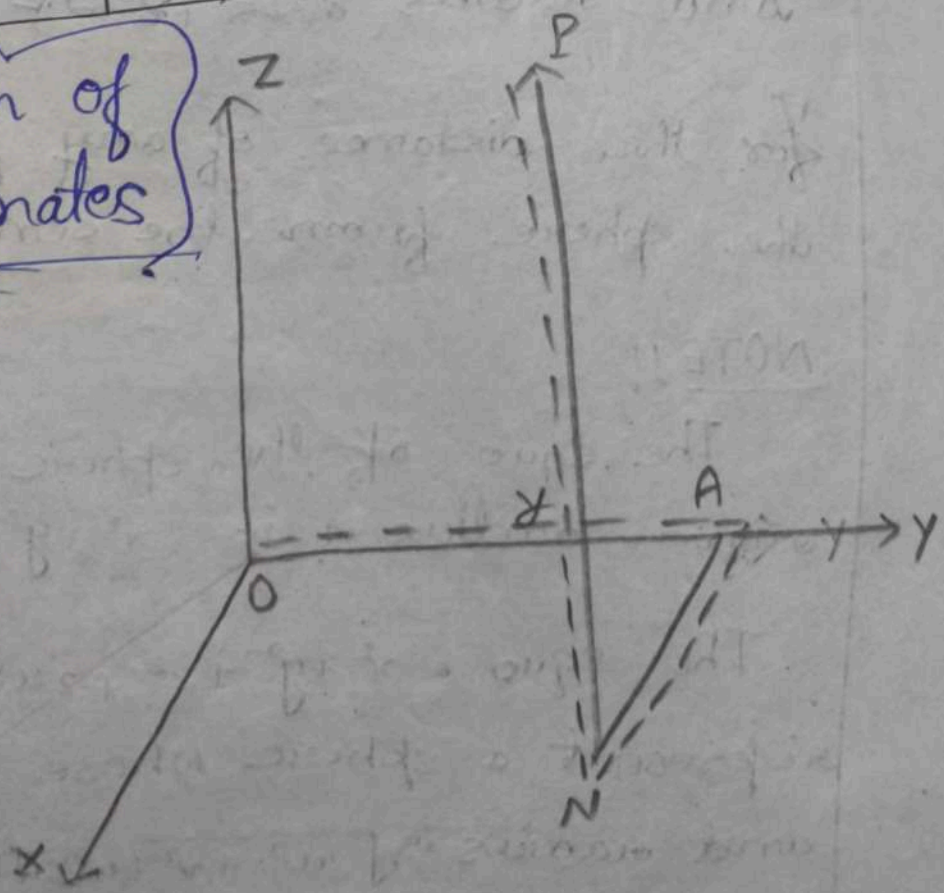
3 - Dimensional

Octants (अष्टांश) (8)

Co-ordinates / Octants	x	y	z
OXYZ	+	+	+
OX'YZ	−	+	+
OXY'Z	+	−	+
OXYZ'	+	+	−
OX'Y'Z	−	−	+
OXY'Z'	+	−	−
OX'YZ'	−	+	−
OX'Y'Z'	−	−	−

Explanation of Co-ordinates



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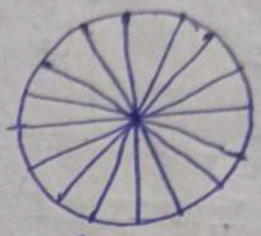
Sphere

* **Sphere :-** A sphere is the locus of a point which remains at a constant distance from a fixed point.



The fixed point called as centre and constant distance and radius of the sphere.

The equ of the sphere whose centre is α, β & γ and radius r is $(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = r^2$.

for the distance of any point $P(x, y, z)$ on the sphere from the centre.

NOTE !!

The equ of the sphere whose centre is origin and its radius a is $x^2 + y^2 + z^2 = a^2$.

The equ $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represent a sphere whose centre is $(-u, -v, -w)$ and radius is $\sqrt{u^2 + v^2 + w^2 - d}$.

⇒ **Sol:-** The equ can be written as.

$$x^2 + 2ux + y^2 + 2vy + z^2 + 2wz = -d$$

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$(x^2+2ux+u^2)+(y^2+2vy+v^2)+(z^2+2wz+w^2) = u^2+v^2+w^2-d.$

$\Rightarrow [x+u]^2+[y+v]^2+[z+w]^2 = \sqrt{u^2+v^2+w^2-d}.$

$\Rightarrow [x-(-u)]^2+[y-(-v)]^2+[z-(-w)]^2 = \left(\sqrt{u^2+v^2+w^2-d}\right)^2.$

Compare it with $\boxed{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=r^2}$

Hence, the centre is $(\alpha,\beta,\gamma) = (-u,-v,-w)$

and radius $r = \sqrt{u^2+v^2+w^2-d}$.

* Characteristic of sphere :-

1. It is 2nd degree equⁿ in (x,y,z) $\Rightarrow x^2+y^2+z^2=r^2$.
2. The coefficient of ~~$x^2+y^2+z^2$~~ $x^2-y^2-z^2$ are required.
3. There are no terms containing xy, yz, zx.

$ax^2+by^2+cz^2=r^2$ is a sphere if

(i) $a \neq b = c$ (ii) $a \neq b \neq c$ (iii) $a=b=c$ (iv) $a=b \neq c$.

Q1. Find the centre & radius of the sphere.

$x^2+y^2+z^2-2x+4y-6z=2.$

$\Rightarrow$ Soln:- $x^2-2x+1+y^2+4y+4+z^2-6z+9 = 2+1+4+9.$

$\Rightarrow (x-1)^2+(y+2)^2+(z-3)^2 = 16.$

$\Rightarrow$ $\therefore$ Centre is $(1,-2,3)$

radius $= \sqrt{1+4+9+2} = \sqrt{16}$

$\Rightarrow \boxed{\text{radius} = 4}$ Δ

Q. Find the equ of the sphere whose centre is $(1,-2,3)$ and radius is 4.

$\Rightarrow$ Soln:- $(x-1)^2+(y+2)^2+(z-3)^2 = 4^2$ Δ

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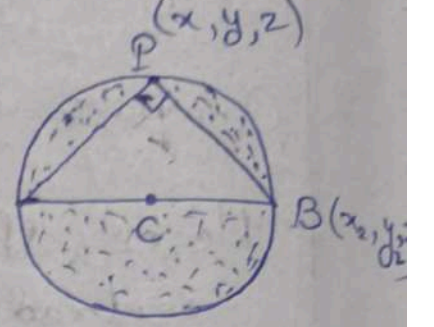
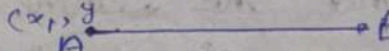
Find the equo of the sphere describe on the join of (x_1, y_1, z_1) & (x_2, y_2, z_2) as a diameter or to find the equo of the sphere, the ends of whose diameter are (x_1, y_1, z_1) & (x_2, y_2, z_2).



⇒ Soln:- Let the two given points be $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ and $P(x, y, z)$ be any points on the sphere.

The plane PAB passes through the centre of the sphere.

Therefore, the section of the sphere by this plane is a great circle whose diameter is AB.



The direction ratios of AP are $(x_2 - x_1), (y_2 - y_1)$ & $(z_2 - z_1)$ resp. l_1, m_1, n_1

and those of BP are l_2, m_2, n_2

$$BP = (x - x_2) \cdot (y - y_2) \quad (z - z_2)$$

Since $\angle APB = \pi/2$, i.e., AP is $\perp^r$ BP

$$\therefore l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

$$\Rightarrow x^2 - x x_1 - x x_2 + x_1 x_2 + y^2 - y y_1 - y y_2 + y_1 y_2 + z^2 - z z_1 - z z_2 + z_1 z_2 =$$

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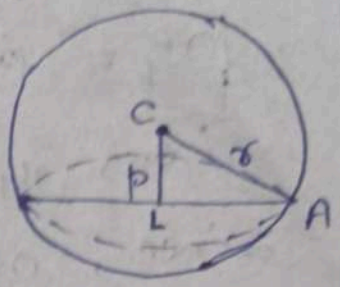
Section of a Sphere by a Plane

It is a circle and a section of sphere by a plane through its centre, is called a great circle.



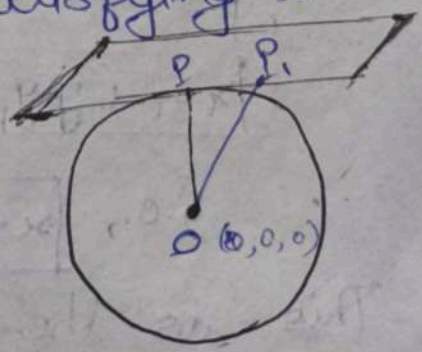
Thus a equo $\boxed{x^2+y^2+z^2+2ux+2vy+2wz+d=0}$ (circle's equo)

and $\boxed{Ax+By+Cz+D=0}$ ⇒ (plane's equo)

taken together a circle (see fig.) having centre L and radius $(LA=\sqrt{r^2-p^2})$.

The equo of any sphere through the circle of intersection of sphere, $\boxed{S=0}$ and the plane $\boxed{P=0}$ is $\boxed{S+kP=0}$

For the equo $\boxed{S+KU=0}$ represents a sphere and the points of intersections of the sphere, $\boxed{S=0}$ and the plane, $\boxed{U=0}$ satisfying it



Equo of the tangent plane

So, that the equo of the tangent at the plane at any point of the sphere $\boxed{x^2+y^2+z^2=a^2}$ is

$$\boxed{xx_1+yy_1+zz_1=a^2}$$

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If $P(x,y,z)$ at any point on tangent plane and direction ratio of the radius

$$\therefore \boxed{OP_1 = (x_1-0), (y_1-0), (z_1-0)}$$

i.e., $(x_1, y_1, z_1) = OP_1$

Since, OP_1 is normal to the tangent plane at P_1, i.e., $\boxed{OP_1 \perp P_1P}$.

$\therefore x_1(x_1-x) + y_1(y_1-y) + z_1(z_1-z) = 0.$

$\Rightarrow x_1^2 - xx_1 + y_1^2 - yy_1 + z_1^2 - zz_1 = 0.$

$\Rightarrow x_1^2 + y_1^2 + z_1^2 = xx_1 + yy_1 + zz_1$

i.e., $\boxed{xx_1 + yy_1 + zz_1 = x_1^2 + y_1^2 + z_1^2}$ ——— (1).

Since, the given sphere $(x^2+y^2+z^2=a^2)$ is passing through the point $P_1(x_1, y_1, z_1)$.

So, $\boxed{x_1^2 + y_1^2 + z_1^2 = a^2}$ ——— (2)

Putting the value of (2) in (1), we get

$$\boxed{xx_1 + yy_1 + zz_1 = a^2}$$

i.e., $\boxed{xx_1 + yy_1 + zz_1 = a^2}$

This is the required equ of the tangent plan

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